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Research Report No. EM-31

ELECTROMAGNETIC SCATTERING MATRICES OF STRATIFIED ANISOTROPIC MEDIA

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ABSTRACT

The problem is treated of an elliptically polarized plane electromagnetic wave incident obliquely on a continuously stratified anisotropic scattering medium. The medium occupies a region of space between two parallel planes ($0 \leq z \leq a$ say) and is characterized by a magnetic susceptibility which is zero and an electric susceptibility which is a continuous matrix function of z. It is required to obtain the reflected and transmitted plane waves on the incident and far sides of the medium, respectively.

The problem is attacked by means of the Schwinger integral equation-variational technique. An integral equation is derived for the electric field vector within the scattering medium. Next, the problem is reduced to computing the elements of a certain "reflection matrix" and "transmission matrix". A set of variational expressions for these matrix elements is then constructed by means of simple considerations in the theory of abstract linear spaces. The variational expressions have the form of functionals depending on two independent vector functions; their Euler equations are the given integral equation and a certain "adjoint" integral equation. Finally, a reciprocity theorem is proved, based on a symmetry property of the variational expressions.

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1. Introduction

The use of variational principles for the formulation of problems in mathematical physics is, of course, very old; one thinks of Hamilton's principle, the principle of least action, and the various kinds of minimum energy principles that have been employed to characterize a multitude of physical situations. The attractiveness of these principles is ascribable to a feature common to all of them, which is that they afford an extremely concise description of the phenomena under investigation. Moreover, in a great many problems the variational principles lead very naturally to practical methods of estimating numerically the quantities it is desired to calculate.

In recent years, the existing wide interest in variational procedures has received a powerful stimulus from the discovery by Julian Schwinger that a broad class of problems in the diffraction or scattering of radiation by obstacles can be formulated variationally.* These problems are of the type in which a known field of radiation impinges on some kind of obstacle and it is required to determine, at least approximately, the spatial distribution of the intensity and phase of the radiation scattered by the obstacle at distances from the latter which are large compared to a wavelength; in mathematical terms, one desires the asymptotic form of the scattered wave at infinity. Schwinger's attack on this problem consists first in deriving an integral equation for the unknown field within or on the obstacle. Then a set of functional expressions is constructed which are stationary with respect to small variations in the function on which they depend, in the neighborhood of solutions to the integral equation, ie., the latter is the Euler equation for the functionals. Finally, it is shown that the stationary values of these functionals are precisely the values of those numerical parameters which characterize the scattered wave at infinity.

It is this last fact which is chiefly responsible for the practical usefulness of Schwinger's method since it enables one to obtain approximations to the far scattered field without becoming unduly involved in the details of the scattering process; one substitutes trial fields into the variational functionals, and if these fields have been skillfully

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* Hulthén, L. (K. Fys. Sälls.Lund Forhand.vol.14,#21 1944) and Ginzburg, V.L. (J.Phys.USSR vol. 7, p.289, 1943) have used alternative variational principles, but we shall not deal with them here.

chosen or perhaps derived by iteration from the integral equation, then the stationary character of the functionals insures a determination of the scattered wave parameters sufficiently accurate for many applications.

Schwinger's procedure has been applied very extensively by him to the problem of the effect of discontinuities in electromagnetic waveguides[1]; particular cases of this problem have also been examined by Miles[2] using Schwinger's method. The question of the diffraction of scalar waves by a perfectly reflecting plane screen and the complementary question of diffraction by a plane obstacle have been dealt with by Levine and Schwinger[3,4,5] by means of the variational formalism, while Miles[6] has generalized this approach to the aperture problem so as to include electromagnetic waves. Sollfrey[7] has employed Schwinger's procedure to formulate the problem of the diffraction of scalar waves by an arbitrarily shaped obstacle and has thereby obtained a formula for the scattering cross-section. The diffraction effect of an infinite plane grating has been treated by Shmoys[8] through the use of Schwinger's method; the grating consists of infinitely many parallel wires of arbitrary cross section, and the waves are electromagnetic. Finally, diFrancia[9] has considered the case of a plane scalar wave obliquely incident upon a plane stratified isotropic medium, and has applied Schwinger's procedure to obtain a variational expression for the reflection coefficient.

In the present paper, we shall show how Schwinger's method can be adapted to a problem which is more complicated in structure than any of the aforementioned ones. Essentially, our problem is a generalization of the one considered by diFrancia; we shall be concerned with the case of an elliptically polarized plane electromagnetic wave with harmonic time dependence, obliquely incident upon a continuously stratified, anisotropic, scattering medium. The medium occupies a region of space between two parallel planes to which the planes of stratification are also parallel. We seek

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9. di Francia, G. Toraldo, Il Nuovo Cimento,vol.7,p.255,May 7,1950(In Italian).

the intensity and polarization of the plane wave reflected back into the region from which the incident wave arrives, as well as the corresponding properties of the transmitted plane wave on the far side of the obstacle. This problem is of great practical importance in connection with the effect of the earth's ionospheric region on the propagation of radio waves since the planar model of the ionosphere can be satisfactorily represented by the kind of inhomogeneous anisotropic medium just described. Indeed, because of its significance for radio-ionosphere work, the problem has received some attention at the hands of applied mathematicians using more conventional methods of analysis[10,11,12], and it is one of the objects of this paper to demonstrate that the Schwinger integral equation-variational technique provides a more succinct treatment of the problem than the methods hitherto employed upon it.

Another, and quite distinct point which we shall make is that it proves highly effective to employ certain elementary considerations in the theory of abstract linear spaces, for the purpose of assisting in the variational formulation. It will be clearly seen how the construction of the variational functionals is expedited by the introduction of these simple abstract considerations.

The arrangement of the material in this paper is as follows: in Section 2 we shall give a precise mathematical statement of our problem. In Section 3, we derive an exact vector integral equation which is satisfied by the unknown electric field within the scattering medium. Following this, a certain "reflection matrix" R and "transmission matrix" T will be defined having the property that they operate on the vector amplitude of the incident electric field to yield the amplitudes of the reflected and transmitted electric field, respectively; hence, our problem reduces to computing the elements of R and T. (If we know the reflected and transmitted electric fields, we can obtain the corresponding magnetic fields from Maxwell's equations.) In Section 5, we shall state and prove Schwinger's fundamental variational principle in the form of a theorem on the stationary property of a certain ratio of inner products in an abstract linear space. Each of the elements of the matrices R and T will then be proved

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10. Rydbeck, O.E.H., On the Propagation of Radio Waves, Trans. Chalmers U. of Tech., No. 34, 1944.
11. Wilkes, M. V., Proc. Royal Soc. A vol. 189, p. 130, 1947.
12. Bremmer, H., Philips Research Rep. vol. 4, p. 189, 1949.

to equal the stationary value of a corresponding functional expression. The form of this expression will be deduced by setting up an identification between the elements of the abstract space and the vector functions of the original problem and inserting the latter into the inner product expression. The variational functionals all depend on two independent vector functions, and will be shown to become stationary when the vector functions are chosen respectively to be the solutions to the integral equation and to a certain "adjoint" integral equation. (Although we have not hitherto mentioned this adjoint equation, its introduction is an essential feature of the Schwinger technique.) Finally, we shall give a physical interpretation of the adjoint equation which in conjunction with a certain symmetry property of the variational forms, will lead to a reciprocity theorem whose significance for the ionosphere will be explained.

To sum up, therefore, the present paper contains a complete formulation of the Schwinger variational procedure as adapted to the scattering problem in question. Actual computations of the matrix elements for specific scattering media representing the earth's ionosphere are to appear in the future.

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2. Statement of Problem

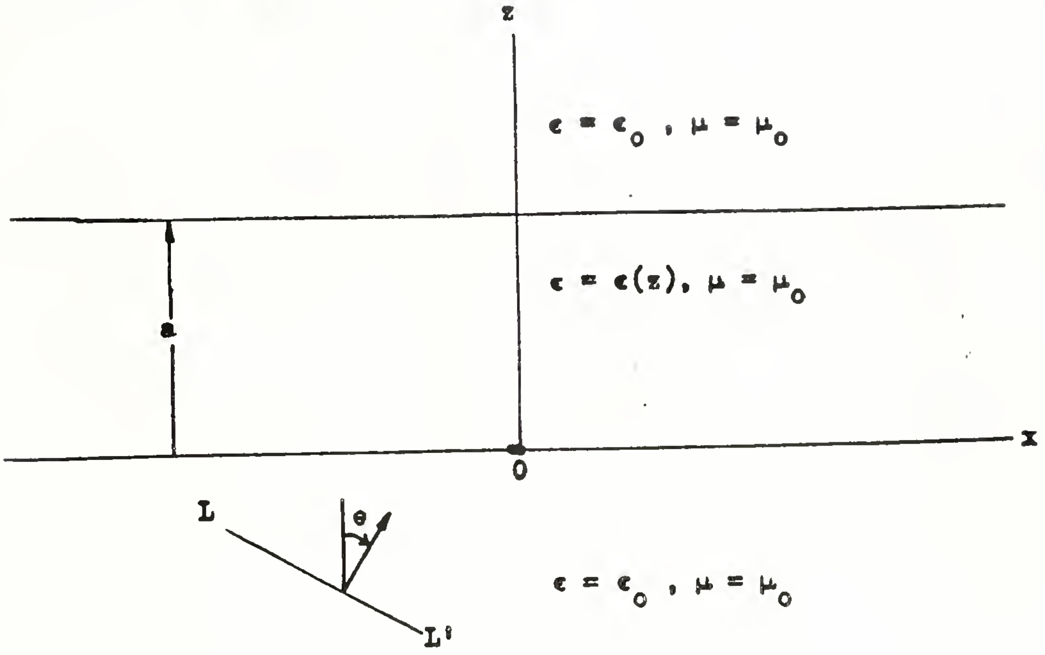
z
$\epsilon = \epsilon_0$, $\mu = \mu_0$
$\epsilon = \epsilon(z)$, $\mu = \mu_0$
a
x
0
L
θ
L'
$\epsilon = \epsilon_0$, $\mu = \mu_0$

Fig. 1

Referring to Fig. 1, 0 is the origin of a right-handed cartesian coordinate system (x,y,z) with the positive y direction into the paper. The regions $z < 0$ and $z > a$ are assumed to be free space with dielectric constant ϵ_0 and magnetic permeability μ_0*. The region $0 \leq z \leq a$ is filled with some material medium having the same magnetic permeability μ_0 but characterized by a complex dielectric constant which must in general be represented by a matrix, $\epsilon(z)$. The elements of this matrix are known, continuous, complex functions of z which depend also on the frequency, although for the sake of brevity, this latter dependence will not be explicitly indicated. (In the special case where $\epsilon(z)$ is independent of frequency, its elements are all real functions.) We shall further require that $\epsilon(0) = \epsilon(a) = \epsilon_0 I$, where I is the unit matrix; hence both the dielectric constant and magnetic permeability are continuous across the boundary planes $z = 0$ and $z = a$.

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* The MKS system is used throughout.

We now suppose that an elliptically polarized plane electromagnetic wave with harmonic time dependence of angular frequency ω impinges on the medium in question from the region $z < 0$. In Fig. 1, the paper is the plane of incidence, L-L' is its intersection with a typical incident plane wavefront, and the angle of incidence is θ, measured clockwise from the positive z direction. We seek the intensities and polarizations of the waves reflected back into the region $z < 0$ and transmitted into the region $z > a$.

To state the problem more completely: let all field vectors have time dependence $e^{-i\omega t}$ and let $\bar{E}$ and $\bar{H}$ represent the spatially dependent parts of the total electric and magnetic fields respectively at any point (x,y,z), where $\bar{E}$ and $\bar{H}$ are to obey the following conditions:

a) $\bar{E}$ and $\bar{H}$ shall be continuous with continuous first derivatives and shall satisfy Maxwell's equations throughout the whole space, i.e.,

1) $$\nabla \times \bar{E} - i\omega\mu\bar{H} = 0$$

2) $$\nabla \times \bar{H} + i\omega\epsilon\bar{E} = 0$$

where $\mu = \mu_o$ everywhere and the matrix ϵ[*] varies as previously described; μ and ϵ are thus everywhere continuous.

b) For $z < 0$, $\bar{E}$ and $\bar{H}$ shall have the forms

$\bar{E}_o e^{ik(x \sin\theta + z\cos\theta)} + \bar{E}_R e^{ik(x\sin\theta - z\cos\theta)}$ and $\bar{H}_o e^{ik(x\sin\theta + z\cos\theta)} + \bar{H}_R e^{ik(x\sin\theta - z\cos\theta)}$

respectively, where $\bar{E}_o$, $\bar{E}_R$, $\bar{H}_o$, and $\bar{H}_R$ are constant complex vector amplitudes and $k = \omega\sqrt{\mu_o\epsilon_o}$. $\bar{E}_o$ and $\bar{H}_o$ are the amplitudes of the incident electric and magnetic fields and are assumed known; $\bar{E}_R$ and $\bar{H}_R$ are the corresponding amplitudes of the reflected fields and are unknown.

c) For $z > a$, $\bar{E}$ and $\bar{H}$ shall have the forms $\bar{E}_T e^{ik(x\sin\theta + z\cos\theta)}$ and $\bar{H}_T e^{ik(x\sin\theta + z\cos\theta)}$ respectively, where $\bar{E}_T$ and $\bar{H}_T$ are the vector amplitudes of the transmitted electric and magnetic fields. Like $\bar{E}_R$ and $\bar{H}_R$, $\bar{E}_T$ and $\bar{H}_T$ are unknown.

Then the specific problem is to obtain $\bar{E}_R$, $\bar{H}_R$, $\bar{E}_T$, and $\bar{H}_T$, being given the vectors $\bar{E}_o$ and $\bar{H}_o$ and the fact that $\bar{E}$ and $\bar{H}$ satisfy conditions (a), (b), and (c). It should be noted, however, that condition (b) is easily

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[*] In equation (2), $\epsilon\bar{E}$ means matrix multiplication of ϵ and $\bar{E}$, where $\bar{E}$ is regarded as a column matrix.

shown to imply that the incident and reflected electromagnetic fields in the region $z < 0$ must separately satisfy Maxwell's equations and that therefore $\overline{E}_0$ and $\overline{H}_0$ cannot be assigned independently of each other but must be so chosen that the incident field obeys equations (1) and (2) in the region $z < 0$.

That the above problem is well set, i.e., that the conditions (a), (b), and (c) are just sufficient to determine $\overline{E}$ and $\overline{H}$ can be seen in a somewhat rough way as follows: from Fig. 1 and the description of the problem, it is clear that $\overline{E}$ and $\overline{H}$ are independent of y. Furthermore, since ϵ as a function of position depends only on z, the assumption that $\overline{E}$ and $\overline{H}$ have an x dependence $e^{ikx\sin\theta}$ within the scattering medium as well as outside it transforms equations (1) and (2) into a system of ordinary differential equations in z within the scattering medium. By simple eliminations, this system can be reduced to a set of four linear first order equations with continuous coefficients in which the dependent variables are the x and y components of $\overline{E}$ and $\overline{H}$. It is well known that such a system always possesses a continuously differentiable solution in which each of the four unknown functions depends linearly on the four constants of integration of the system; these four constants are of course to be determined.

Now we have seen that outside the scattering medium the reflected and transmitted fields must individually satisfy Maxwell's equations, so that of the four vectors $\overline{H}_R$, $\overline{H}_T$, $\overline{E}_R$, and $\overline{E}_T$, only the last two, for example need be solved for. Moreover, the reflected and transmitted fields have been assumed to be plane waves, for which it is known(13) that $\overline{E}_R$ and $\overline{E}_T$ must lie in the planes of their respective wavefronts. Therefore $\overline{E}_R$ and $\overline{E}_T$ can possess only two independent components each. Thus the field outside the scattering medium, like the field within it, is a linear function of four independent unknown constants, making a total of eight unknown constants to be determined in the problem. Since the field vectors are to be everywhere continuous, however, imposition of the continuity requirement on the x and y components of $\overline{E}$ and $\overline{H}$ at the boundary planes $z = 0$ and $z = a$ leads to a non-homogeneous* system of eight linear algebraic equations for the eight unknowns. The determinant of the system can be shown not to vanish, and therefore there is one and only one solution.

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13. Stratton, J. A. Electromagnetic Theory p.272-273.

* The system is non-homogeneous because $\overline{E}_0$ and $\overline{H}_0$ are known.

3. Derivation of the Integral Equation

In the present section we shall derive the fundamental vector integral equation for the electric field in the scattering region $0 \leq z \leq a$, this equation constituting the actual starting point of the theory. Specifically, we shall show that if $\bar{E}$ and $\bar{H}$ satisfy conditions (a), (b), and (c) of Section 2, then the z dependent part of $\bar{E}$ must be a solution of the integral equation.

The first step in our procedure is to construct a Hertz potential vector to which $\bar{E}$ and $\bar{H}$ are related by a pair of differentiation formulas, and to this end we employ the following principle* in electromagnetic theory: let the vectors $\bar{f}$ and $\bar{g}$ be any solution to Maxwell's equations in a region where $\mu = \mu_0$ and ϵ is a continuous but otherwise arbitrary matrix function of position, i.e.,

$$3) \qquad \nabla \times \bar{f} - i\omega\mu_0 \bar{g} = 0$$

$$4) \qquad \nabla \times \bar{g} + i\omega\epsilon \bar{f} = 0$$

Then there always exists at least one vector function $\bar{\phi}$ satisfying the equation

$$5) \qquad \nabla^2 \bar{\phi} + k^2 \bar{\phi} = - \frac{1}{\epsilon_0} \left[\epsilon - \epsilon_0 I \right] \bar{f}$$

and having the further property that

$$6) \qquad \bar{f} = \nabla(\nabla \cdot \bar{\phi}) + k^2 \bar{\phi}$$

and

$$7) \qquad \bar{g} = -i\omega\epsilon_0 \nabla \times \bar{\phi}$$

In equation (5), ∇^2 represents the ordinary Laplacian operating on each of the rectangular components of $\bar{\phi}$, k is $\omega\sqrt{\mu_0\epsilon_0}$ as before, and I is the unit matrix. $\bar{\phi}$ is called the Hertz potential of the electromagnetic field defined by $\bar{f}$ and $\bar{g}$.

The foregoing principle evidently assures us that corresponding to the pair of vectors $\bar{E}(x,z)$ and $\bar{H}(x,z)$ obeying conditions (a), (b), and (c) of Section 2, there always exists at least one function $\bar{\psi}(x,z)$ say, satisfying the equation

$$8) \qquad \nabla^2 \bar{\psi}(x,z) + k^2 \bar{\psi}(x,z) = - \frac{1}{\epsilon_0} \left[\epsilon(z) - \epsilon_0 I \right] \bar{E}(x,z)$$

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* This principle is proved in Appendix I.

for all x and z, and to which $\bar{E}$ and $\bar{H}$ are further related by the formulas

$$9)\qquad \bar{E}(x,z) = \nabla\,[\nabla \cdot \bar{\psi}(x,z)] + k^2\,\bar{\psi}(x,z)$$

$$10)\qquad \bar{H}(x,z) = -i\omega\epsilon_0\,\nabla \times \bar{\psi}(x,z)$$

for all x and z.

We shall now show that the existence of a function $\bar{\psi}$ which satisfies equations (8), (9), and (10) necessarily leads to the determination of its form, and that from the form of $\bar{\psi}$ we can deduce that the z dependent part of $\bar{E}(x,z)$ satisfies an integral equation.

To begin with, consider the function $\bar{\psi}$ outside the scattering region. There, the right side of equation (8) must vanish since $\epsilon(z) = \epsilon_0 I$, and hence conditions (b) and (c) on $\bar{E}(x,z)$ and $\bar{H}(x,z)$ together with equations (9) and 10) imply that

$$11)\qquad \bar{\psi}(x,z) = \bar{\psi}_0\, e^{ik(x\sin\theta + z\cos\theta)} + \bar{\psi}_R\, e^{ik(x\sin\theta - z\cos\theta)}, \qquad z < 0$$

$$12)\qquad \bar{\psi}(x,z) = \bar{\psi}_T\, e^{ik(x\sin\theta + z\cos\theta)} \qquad z > a$$

where $\bar{\psi}_0$, $\bar{\psi}_R$, and $\bar{\psi}_T$ are constant complex vector amplitudes.

With the aid of equations (11) and (12), which express the form of $\bar{\psi}$ outside the scattering region, we can, in the following manner, obtain an expression for $\bar{\psi}$ which is valid anywhere.

Remembering that the x dependence of $\bar{E}$ and $\bar{H}$ is $e^{ikx\sin\theta}$, it follows that $\bar{\psi}$ must have the same x dependence. If in equation (8) we denote the vector $[\epsilon(z) - \epsilon_0 I]\,\bar{E}(x,z)$ by $\bar{P}(x,z)$, the introduction of the x dependence $e^{ikx\sin\theta}$ reduces the equation to

$$13)\qquad \frac{d^2\bar{\psi}(z)}{dz^2} + k^2\cos^2\theta\,\bar{\psi}(z) = -\frac{1}{\epsilon_0}\,\bar{P}(z)$$

where $\bar{\psi}(z)$ and $\bar{P}(z)$ are the z dependent parts of $\bar{\psi}(x,z)$ and $\bar{P}(x,z)$.

Equation (13) of course represents three separate scalar equations, each of which involves only one of the three components of $\bar{\psi}(z)$ and $\bar{P}(z)$. Therefore, we may introduce a scalar Green's function[14] G of the form

$$14)\qquad G(z,z') = \frac{e^{ik|z-z'|\cos\theta}}{2\,ik\cos\theta}$$

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[14] Courant-Hilbert, Method. der Math. Phys.I,p.302.

and apply Green's formula[15] to $\bar{\psi}(z)$ and $G(z,z')$ over the interval $-\infty \leq z \leq \infty$.

There results, after changing the independent variable from z to z' in equation (13),

$$15)\quad -\frac{1}{\epsilon_0}\int_{-\infty}^{\infty} G(z,z')\bar{P}(z')dz' = \left(\frac{d\bar{\psi}(z')}{dz'}G(z,z') - \frac{\partial G(z,z')}{\partial z'}\bar{\psi}(z')\right)\Bigg|_{z'=-\infty}^{z'=z-0}$$

$$+\left(\frac{d\bar{\psi}(z')}{dz'}G(z,z') - \frac{\partial G(z,z')}{\partial z'}\bar{\psi}(z')\right)\Bigg|_{z'=z+0}^{z'=\infty}$$

At $z'=-\infty$, the form of $\bar{\psi}(z')$ is evidently given by the z dependent part of expression (11) while the form of G is $\frac{e^{ik(z-z')\cos\theta}}{2ik\cos\theta}$. Insertion of these expressions into the first term on the right of equation (15) gives as the lower limit of that term

$$16)\quad -\left[ik\cos\theta\left(\bar{\psi}_0 e^{ikz'\cos\theta} - \bar{\psi}_R e^{-ikz'\cos\theta}\right)\frac{e^{ik(z-z')\cos\theta}}{2\,ik\cos\theta} + \frac{1}{2}e^{ik(z-z')\cos\theta}\left(\bar{\psi}_0 e^{ikz'\cos\theta} + \bar{\psi}_R e^{-ikz'\cos\theta}\right)\right]$$

which reduces simply to

$$17)\qquad -\bar{\psi}_0\, e^{ikz\cos\theta}$$

Similarly, at $z'=+\infty$, $\bar{\psi}(z')$ behaves like the z dependent part of expression (12) while G takes the form $\frac{e^{ik(z'-z)\cos\theta}}{2\,ik\cos\theta}$. Insertion of these expressions into the second term on the right of equation (15) gives as the upper limit of that term

$$18)\quad ik\cos\theta\,\bar{\psi}_T e^{ikz'\cos\theta}\frac{e^{ik(z'-z)\cos\theta}}{2\,ik\cos\theta} - \frac{1}{2}e^{ik(z'-z)\cos\theta}\bar{\psi}_T\, e^{ikz'\cos\theta} = 0.$$

Due to the continuity of $G(z,z')$ at $z'=z$, the terms on the right of equation (15) involving $\frac{d\bar{\psi}}{dz'}$ must cancel at $z'=z$, leaving only

$$19)\quad -\frac{\partial G(z,z')}{\partial z'}\bar{\psi}(z')\Bigg|_{z'=z+0}^{z'=z-0} = -\left[-\frac{1}{2}e^{ik(z-z')\cos\theta}\bar{\psi}(z') - \frac{1}{2}e^{ik(z'-z)\cos\theta}\bar{\psi}(z')\right]_{z'=z} = \bar{\psi}(z)$$

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[15] Ibid., p. 237.

Thus equation (15) may now be written

$$20)\qquad -\frac{1}{\epsilon_0}\int_{-\infty}^{\infty} G(z,z')\,\bar{P}(z')\,dz' = -\bar{\psi}_0\, e^{ikz\cos\theta} + \bar{\psi}(z)$$

It follows from its definition, however, that the vector $\bar{P}$ vanishes outside the region $0 \leq z \leq a$. Therefore, the integration in equation (20) need extend only from 0 to a so that the equation becomes

$$21)\quad \bar{\psi}(z) = \bar{\psi}_0 e^{ikz\cos\theta} - \frac{1}{2i\epsilon_0 k\cos\theta}\int_0^a e^{ik|z-z'|\cos\theta}\,\bar{P}(z')\,dz'$$

where we have replaced G by its explicit form, given by equation (14).

Multiplication of both sides of equation (21) by $e^{ikx\sin\theta}$ results in

$$22)\qquad \bar{\psi}(x,z) = \bar{\psi}_0 e^{ik(x\sin\theta + z\cos\theta)} - \frac{e^{ikx\sin\theta}}{2i\epsilon_0 k\cos\theta}\int_0^a e^{ik|z-z'|\cos\theta}\,\bar{P}(z')\,dz'$$

One further necessary condition on the form of $\bar{\psi}(x,z)$ can be deduced from condition (b) of Section 2 and equations (9) and (11). It is that $\bar{\psi}_0$, which we have not yet specified, must be such that

$$23)\qquad \left[\nabla(\nabla\cdot\,) + k^2\right]\bar{\psi}_0\, e^{ik(x\sin\theta + z\cos\theta)} = \bar{E}_0\, e^{ik(x\sin\theta + z\cos\theta)}$$

A breakdown of equation (23) into its scalar components together with the fact that $\bar{E}_0$ must lie in the plane of the incident wavefront shows at once that $\bar{\psi}_0$ can always be chosen so as to satisfy equation (23), and indeed in an infinite number of ways.

Our results at this point may be summed up as follows: if $\bar{E}(x,z)$ and $\bar{H}(x,z)$ obey conditions (a), (b), and (c) of Section 2, then the vector $\bar{\psi}(x,z)$ related to $\bar{E}(x,z)$ and $\bar{H}(x,z)$ through equations (8), (9), and (10) must necessarily be of the form given by equation (22) for all x and z, in which $\bar{\psi}_0$ is any vector satisfying equation (23).

We may now proceed directly to the deduction of the integral equation as a necessary consequence of the result just stated. Applying the operator $\left[\nabla(\nabla\cdot\,) + k^2\right]$ to both sides of equation (22), we obtain in view of equations (9) and (23)

$$24)\ \bar{E}(x,z) = \bar{E}_0 e^{ik(x\sin\theta + z\cos\theta)} - \left[\nabla(\nabla\cdot\,) + k^2\right]\frac{e^{ikx\sin\theta}}{2i\epsilon_0 k\cos\theta}\int_0^a e^{ik|z-z'|\cos\theta}\,\bar{P}(z')\,dz'$$

where the effect of the operator on the second term on the right remains to be worked out. This calculation, although elementary, is rather complicated and we have therefore relegated it to Appendix II. The final result is

25) $$\left[\nabla(\nabla\cdot) + k^2\right]\frac{e^{ikx\sin\theta}}{2i\epsilon_0 k\cos\theta}\int_0^a e^{ik|z-z'|\cos\theta}\,\bar{P}(z')dz' =$$

$$-\frac{ike^{ikx\sin\theta}}{2\epsilon_0\cos\theta}\int_0^a e^{ik|z-z'|\cos\theta}A(z,z')\bar{P}(z')dz' + \frac{1}{\epsilon_0}e^{ikx\sin\theta}P_3(z)\underline{k}$$

in which $A(z,z')$ is a matrix defined by

26) $$A(z,z') = \begin{pmatrix} \cos^2\theta & 0 & -\sin\theta\cos\theta \\ 0 & 1 & 0 \\ -\sin\theta\cos\theta & 0 & \sin^2\theta \end{pmatrix} \quad z' < z$$

27) $$A(z,z') = \begin{pmatrix} \cos^2\theta & 0 & \sin\theta\cos\theta \\ 0 & 1 & 0 \\ \sin\theta\cos\theta & 0 & \sin^2\theta \end{pmatrix} \quad z' > z$$

and in which $A\bar{P}$ in the integral means matrix multiplication of A and $\bar{P}$ regarding the vector $\bar{P}$ as a column matrix, $P_3(z)$ is the z component of $\bar{P}(z)$, and $\underline{k}$ is a unit vector in the positive z direction.

Substituting for the second term on the right of equation(24) the evaluation as given by equation (25), we get after dropping the x dependence

28) $$\bar{E}(z) + \frac{1}{\epsilon_0}P_3(z)\,\underline{k} = \bar{E}_0\,e^{ikz\cos\theta} + \frac{ik}{2\epsilon_0\cos\theta}\int_0^a e^{ik|z-z'|\cos\theta}A(z,z')\bar{P}(z')dz'$$

It is convenient to introduce two new matrices, B(z) and C(z). B(z) is defined through the relationship

29) $$\epsilon_0 B(z) = \epsilon(z) - \epsilon_0 I$$

and therefore B(z) vanishes outside the region $0 \leq z \leq a$. From the definition of $\bar{P}(x,z)$ we see that

30) $$\bar{P}(z) = \epsilon_0 B(z)\,\bar{E}(z)$$

B(z) is the so-called electric susceptibility matrix of the scattering medium, while $\bar{P}$ is known as the electric polarization vector of the medium.

The matrix C(z) is defined as having its first two rows zero and its third row equal to the third row of B(z). Evidently then

31) $$P_3(z)\,\underline{k} = \epsilon_0 C(z)\,\bar{E}(z)$$

Substituting equations (30) and (31) into equation (28), we finally obtain

$$32)\quad \left[I + C(z) \right] \bar{E}(z) = \bar{E}_0 e^{ik z \cos \Theta} + \frac{ik}{2 \cos \Theta} \int_0^a e^{ik|z-z'|\cos \Theta} A(z,z') B(z') \bar{E}(z') dz'$$

Equation (32) is the desired vector integral equation for the z dependent part of the electric field in the scattering region $0 \leq z \leq a$. Together with the adjoint integral equation referred to in the Introduction and to be derived presently, it plays a fundamental role in the construction of the variational formulas for the elements of the reflection and transmission matrices.

There are several points regarding equation (32) which we now wish to mention briefly. First, although the integration extends only over the scattering region, the method of deriving the equation implies that it correctly represents the electric field anywhere; in fact, the equation has the clear physical interpretation that it expresses the total electric field at any point as the sum of the incident field and a field produced by the integrated effect of induced sources within the scattering medium.

Second, equation (32) is a system of three scalar integral equations for the x, y, and z components of $\bar{E}(z)$ and as such it possesses a well established existence theory[16,17]. This theory assures us that the equation has a unique, continuously differentiable solution, because the kernel is continuous except for the finite jump in $A(z,z')$ at $z = z'$. Now we have just finished showing that every electric field obeying the conditions of our original problem provides a solution to equation (32), and the uniqueness theorem for the integral equation now implies that the converse statement is also true, viz., if we start with equation (32) in which the first term on the right is the z dependent part of a plane wave solution to the free space Maxwell's equations, then the solution to the equation furnishes an electric field satisfying all the conditions of our original problem. Thus the integral equation and the original problem are fully equivalent.

Finally, it should be noted that the matrix $A(z,z')$ is singular; the significance of this fact will become more apparent in the next section, dealing with the reflection and transmission matrices, where we shall deduce several important and useful properties of $A(z,z')$ associated with its singular character.

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16) Hellinger and Toeplitz, Encyklop.Math.Wiss. Bd. 2, Teil 3, Heft 9.
17) Platrier, Ch. Journal de Math. Series 6, vol.9, p.257 ff. 1913.

4. The Reflection and Transmission Matrices

In the Introduction, we alluded to the fact that the problem of obtaining the complex vector amplitudes $\bar{E}_R$ and $\bar{E}_T$ (and thereby automatically $\bar{H}_R$ and $\bar{H}_T$) can be reduced to that of calculating the elements of a certain "reflection matrix" and a certain "transmission matrix" which operate on $\bar{E}_o$ to yield $\bar{E}_R$ and $\bar{E}_T$ respectively. In this section we shall carry out the reduction and relate these matrices to the integral equation (32).

We begin by considering any two unit orthogonal vectors, $\bar{u}_1$ and $\bar{u}_2$, in the plane of the incident wavefront. Observing once again that for electromagnetic plane waves in free space, the electric and magnetic vectors lie in the plane of the wavefront, we may evidently write

33) $$\bar{E}_o = \mathcal{E}_o^{(1)}\,\bar{u}_1 + \mathcal{E}_o^{(2)}\,\bar{u}_2$$

where $\mathcal{E}_o^{(1)}$ and $\mathcal{E}_o^{(2)}$ are complex scalars.

Similarly, if $\bar{u}_1'$ and $\bar{u}_2'$ are any two unit orthogonal vectors in the plane of the reflected wavefront, we have

34) $$\bar{E}_R = \mathcal{E}_R^{(1)}\bar{u}_1' + \mathcal{E}_R^{(2)}\bar{u}_2'$$

where $\mathcal{E}_R^{(1)}$ and $\mathcal{E}_R^{(2)}$ are complex scalars.

Finally, since the planes of the transmitted and incident wavefronts are parallel, it is convenient to choose the vectors $\bar{u}_1$ and $\bar{u}_2$ of equation (33) as unit orthogonal vectors in the transmitted wavefront. There follows

35) $$\bar{E}_T = \mathcal{E}_T^{(1)}\,\bar{u}_1 + \mathcal{E}_T^{(2)}\,\bar{u}_2$$

where of course $\mathcal{E}_T^{(1)}$ and $\mathcal{E}_T^{(2)}$ are complex scalars.

The original problem posed by conditions (a), (b), and (c) of Section 2 is linear; hence the effect of multiplying $\bar{E}_o$ by a complex number is simply to multiply $\bar{E}_R$ and $\bar{E}_T$ by the same number, while the effect of superimposing two incident waves in the original problem is to superimpose the corresponding reflected and transmitted waves. These consequences of linearity then lead us immediately to the following relationships:

36) $$\mathcal{E}_R^{(1)} = R_{11}\,\mathcal{E}_o^{(1)} + R_{12}\,\mathcal{E}_o^{(2)}$$

37) $$\mathcal{E}_R^{(2)} = R_{21}\,\mathcal{E}_o^{(1)} + R_{22}\,\mathcal{E}_o^{(2)}$$

38) $$\mathcal{E}_T^{(1)} = T_{11}\,\mathcal{E}_o^{(1)} + T_{12}\,\mathcal{E}_o^{(2)}$$

39) $$\mathcal{E}_T^{(2)} = T_{21}\,\mathcal{E}_o^{(1)} + T_{22}\,\mathcal{E}_o^{(2)}$$

where the R_{ij} and T_{ij} are complex scalars. In matrix form, equations (36) and (37) may be written as

$$40) \qquad \bar{E}_R = R\,\bar{E}_o$$

and equations (38) and (39) as

$$41) \qquad \bar{E}_T = T\,\bar{E}_o$$

The matrix $R = \|R_{ij}\|$ is the so-called reflection matrix of the scattering medium, while $T = \|T_{ij}\|$ is the transmission matrix of the medium. In addition to depending on the specific nature of the scattering medium as characterized by its susceptibility matrix $B(z)$, the R_{ij} and T_{ij} are functions of the angle of incidence, Θ, and the frequency, ω. Obviously, a knowledge of the elements R_{ij} and T_{ij} constitutes the solution to our problem; thus, the reflection and transmission matrices contain all the desired information relevant to the effect of the scattering medium on the incident wave.

Now our eventual object with regard to the elements R_{ij} and T_{ij} is the derivation of a variational principle for them, but in order to accomplish this, we must first relate them to certain particular solutions of the integral equation which correspond to special forms of the incident wave. We begin by examining the integral equation (32) for $z < 0$ and $z > a$, i.e., outside the scattering region. Taking first the case of $z < 0$, we see that $C(z) = 0$ and $|z-z'| = (z'-z)$, while $A(z,z')$ takes the form of the constant matrix given in equation (27), which we denote by A_+. Equation (32) then becomes

$$42) \qquad \bar{E}(z) = \bar{E}_o\, e^{ikz\cos\Theta} + e^{-ikz\cos\Theta}\,\frac{ik}{2\cos\Theta}\int_0^a e^{ikz'\cos\Theta} A_+ B(z')\bar{E}(z')dz', \quad z < 0$$

From the form of the right side of equation (42) together with the consideration of condition (b) of Section 2, it is clear that we may write

$$43) \qquad \bar{E}_R = \frac{ik}{2\cos\Theta}\int_0^a e^{ikz'\cos\Theta} A_+\, B(z')\bar{E}(z')\,dz'$$

For the case $z > a$, we again have $C(z) = 0$, while $|z-z'| = (z-z')$ and $A(z,z')$ takes the form of the constant matrix given in equation (26), which we denote by A_-. Equation (32) then becomes

$$44) \qquad \bar{E}(z) = \bar{E}_o\, e^{ikz\cos\Theta} + e^{ikz\cos\Theta}\,\frac{ik}{2\cos\Theta}\int_0^a e^{-ikz'\cos\Theta} A_- B(z')\bar{E}(z')dz', \quad z > a$$

and from the form of the right side of equation (44) together with condition (c) of Section 2, we get

45) $$\bar{E}_T = \bar{E}_o + \frac{ik}{2\cos\theta}\int_0^a e^{-ikz'\cos\theta} A_- B(z')\,\bar{E}(z')\,dz'$$

Formulas (43) and (45) express the vector amplitudes of the reflected and transmitted waves in terms of the electric field in the scattering region $0 \le z \le a$.

Now let $\bar{E}_{(j)}(z)$, $(j = 1,2)$, represent those solutions of the integral equation (32) which correspond respectively to the incident waves of vector amplitude $\bar{E}_{o(j)} = \bar{u}_j$, $(j = 1,2)$, and let $\bar{E}_{R(j)}$ and $\bar{E}_{T(j)}$ represent the corresponding reflected and transmitted wave amplitudes. Then according to equation (43)

46) $$\bar{E}_{R(j)} = \frac{ik}{2\cos\theta}\int_0^a e^{ikz'\cos\theta} A_+ B(z')\,\bar{E}_{(j)}(z')\,dz' \qquad (j=1,2)$$

and according to equation (45)

47) $$\bar{E}_{T(j)} = \bar{u}_j + \frac{ik}{2\cos\theta}\int_0^a e^{-ikz'\cos\theta} A_- B(z')\,\bar{E}_{(j)}(z')\,dz' \qquad (j=1,2)$$

From equation (33) we have

48) $$\bar{E}_{o(j)} = \mathcal{E}_{o(j)}^{(1)}\,\bar{u}_1 + \mathcal{E}_{o(j)}^{(2)}\,\bar{u}_2 \qquad (j=1,2)$$

where from the definition of $\bar{E}_{o(j)}$ it is evident that

49) $$\mathcal{E}_{o(j)}^{(k)} = \delta_{kj} = \begin{matrix} 1, & k = j \\ 0, & k \neq j \end{matrix} \qquad \begin{matrix} (k=1,2) \\ (j=1,2) \end{matrix}$$

Substitution of this last relation into equations (36) and (37) gives

50) $$\mathcal{E}_{R(j)}^{(i)} = \sum_{k=1}^{2} R_{ik}\,\delta_{kj} = R_{ij} \qquad \begin{matrix} (i=1,2) \\ (j=1,2) \end{matrix}$$

and since equation (34), when $\bar{E}_R = \bar{E}_{R(j)}$, becomes

51) $$\bar{E}_{R(j)} = \mathcal{E}_{R(j)}^{(1)}\,\bar{u}_1' + \mathcal{E}_{R(j)}^{(2)}\,\bar{u}_2' \qquad (j=1,2)$$

it follows from equation (50) and the orthogonality of the vectors $\bar{u}_1'$ and $\bar{u}_2'$ that

52) $$\bar{E}_{R(j)} \cdot \bar{u}_i' = R_{ij} \qquad \begin{matrix} (i=1,2) \\ (j=1,2) \end{matrix}$$

Similarly, if we substitute equation (49) into equations (38) and (39), we get

53) $$\mathcal{E}^{(i)}_{T(j)} = \sum_{k=1}^{2} T_{ik}\,\delta_{kj} = T_{ij} \qquad \begin{array}{l}(i=1,2)\\(j=1,2)\end{array}$$

and since equation (35), when $\bar{E}_T = \bar{E}_{T(j)}$, becomes

54) $$\bar{E}_{T(j)} = \mathcal{E}^{(1)}_{T(j)}\,\bar{u}_1 + \mathcal{E}^{(2)}_{T(j)}\,\bar{u}_2 \qquad (j=1,2)$$

it follows from equation (53) and the orthogonality of $\bar{u}_1$ and $\bar{u}_2$ that

55) $$\bar{E}_{T(j)} \cdot \bar{u}_i = T_{ij} \qquad \begin{array}{l}(i=1,2)\\(j=1,2)\end{array}$$

Equations (52) and (55) are significant because they express the essential meaning of the reflection and transmission matrix elements. Thus we see that R_{ij} represents the component in the direction $\bar{u}_i'$ of the vector amplitude of that reflected wave which was excited by an incident wave of vector amplitude $\bar{u}_j$; naturally, T_{ij} has, mutatis mutandis, the same meaning.

If we now take the scalar product of equation (46) with $\bar{u}_i'$, we obtain, in view of equation (52)

56) $$R_{ij} = \frac{ik}{2\cos\theta}\int_0^a e^{ikz'\cos\theta}\,\bar{u}_i' \cdot A_+\,B(z')\,\bar{E}_{(j)}(z')\,dz' \,. \qquad \begin{array}{l}(i=1,2)\\(j=1,2)\end{array}$$

Likewise, the scalar product of equation (47) with $\bar{u}_i$ produces, in conjunction with equation (55)

57) $$T_{ij} = \delta_{ij} + \frac{ik}{2\cos\theta}\int_0^a e^{-ikz'\cos\theta}\,\bar{u}_i \cdot A_-\,B(z')\,\bar{E}_{(j)}(z')\,dz' \qquad \begin{array}{l}(i=1,2)\\(j=1,2)\end{array}$$

It is possible to effect a useful simplification of formulas (56) and (57) by means of the following property of the matrices A_+ and A_- : Every vector in the plane of the reflected wavefront is an eigenvector of A_+ with eigenvalue 1, while every vector in the plane of the incident or transmitted wavefront is an eigenvector of A_- with eigenvalue 1.

To prove this statement, let $\bar{F}$ be any vector in the plane of the feflected wavefront. Then in terms of its rectangular components, $\bar{F}$ must clearly be of the form

58) $$\bar{F} = F_1\cos\theta\,\underline{i} + F_2\,\underline{j} + F_1\sin\theta\,\underline{k}$$

where F_1 and F_2 are arbitrary complex scalars. The product $A_+\bar{F}$ gives

59)
$$A_+\bar{F} = \begin{pmatrix} \cos^2\theta & 0 & \sin\theta\cos\theta \\ 0 & 1 & 0 \\ \sin\theta\cos\theta & 0 & \sin^2\theta \end{pmatrix} \begin{pmatrix} F_1\cos\theta \\ F_2 \\ F_1\sin\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\theta\, F_1\cos\theta + \sin\theta\cos\theta\, F_1\sin\theta \\ F_2 \\ \sin\theta\cos\theta\, F_1\cos\theta + \sin^2\theta\, F_1\sin\theta \end{pmatrix} = \begin{pmatrix} F_1\cos\theta \\ F_2 \\ F_1\sin\theta \end{pmatrix} = \bar{F}$$

which proves the statement for A_+ .

Similarly, let $\bar{G}$ be any vector in the incident wavefront. Then $\bar{G}$ may be written

60)
$$\bar{G} = G_1\cos\theta\,\underline{i} + G_2\,\underline{j} - G_1\sin\theta\,\underline{k}$$

where G_1 and G_2 are arbitrary complex scalars, while the product $A_-\bar{G}$ becomes

61)
$$A_-\bar{G} = \begin{pmatrix} \cos^2\theta & 0 & -\sin\theta\cos\theta \\ 0 & 1 & 0 \\ -\sin\theta\cos\theta & 0 & \sin^2\theta \end{pmatrix} \begin{pmatrix} G_1\cos\theta \\ G_2 \\ -G_1\sin\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\theta\, G_1\cos\theta - \sin\theta\cos\theta\,(-G_1\sin\theta) \\ G_2 \\ -\sin\theta\cos\theta\, G_1\cos\theta + \sin^2\theta(-G_1\sin\theta) \end{pmatrix} = \begin{pmatrix} G_1\cos\theta \\ G_2 \\ -G_1\sin\theta \end{pmatrix} = \bar{G}$$

Thus, the statement is proved. We apply this result by noting first that it implies

62) $$A_+\bar{u}_i' = \bar{u}_i' \qquad (i=1,2)$$

63) $$A_-\bar{u}_i = \bar{u}_i \qquad (i=1,2)$$

Now A_+ and A_- are both symmetric matrices, from which it follows at once that

64) $$\bar{u}_i' \cdot A_+B(z')\,\bar{E}_{(j)}(z') = A_+\bar{u}_i' \cdot B(z')\,\bar{E}_{(j)}(z')$$

65) $$\bar{u}_i \cdot A_-B(z')\,\bar{E}_{(j)}(z') = A_-\bar{u}_i \cdot B(z')\,\bar{E}_{(j)}(z')$$

whence from equations (62) and (63) we obtain

66) $$R_{ij} = \frac{ik}{2\cos\theta}\int_0^a e^{ikz'\cos\theta}\,\bar{u}_i' \cdot B(z')\,\bar{E}_{(j)}(z')\,dz' \qquad \begin{matrix}(i=1,2)\\(j=1,2)\end{matrix}$$

67) $$T_{ij} = \delta_{ij} + \frac{ik}{2\cos\theta}\int_0^a e^{-ikz'\cos\theta}\,\bar{u}_i \cdot B(z')\,\bar{E}_{(j)}(z')\,dz' \qquad \begin{matrix}(i=1,2)\\(j=1,2)\end{matrix}$$

Equations (66) and (67) constitute the desired form of the relationship, already referred to, between the particular solutions $\bar{E}_{(j)}(z)$ of the integral equation (32) and the elements of the reflection and transmission matrices. Although these relations may themselves be used for approximation of the R_{ij} and T_{ij} by replacement of $\bar{E}_{(j)}(z)$ in the integrals with suitable approximate trial functions, their chief importance for us will be in the fundamental role they play in the derivation of the variational formulas.

Before proceeding with the derivation, to be carried out in the next section, we wish to interpret the singular character of the matrix $A(z,z')$. For this purpose we need a further property of the matrices A_+ and A_-, viz., every vector perpendicular to the reflected wavefront is an eigenvector of A_+ with eigenvalue 0, while the corresponding statement holds for A_- and the incident wavefront.

To verify this assertion, let $\bar{V}$ be any vector perpendicular to the reflected wavefront. We may write $\bar{V}$ as

68) $$\bar{V} = V\sin\theta\,\underline{i} - V\cos\theta\,\underline{k}$$

where V is any complex scalar. Then

69) $$A_+\bar{V} = \begin{pmatrix} \cos^2\theta & 0 & \sin\theta\cos\theta \\ 0 & 1 & 0 \\ \sin\theta\cos\theta & 0 & \sin^2\theta \end{pmatrix}\begin{pmatrix} V\sin\theta \\ 0 \\ -V\cos\theta \end{pmatrix} = \begin{pmatrix} \cos^2\theta\, V\sin\theta - \sin\theta\cos\theta\, V\cos\theta \\ 0 \\ \sin\theta\cos\theta\, V\sin\theta - \sin^2\theta\, V\cos\theta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

and in an exactly analogous way we can show that

70) $$A_-\bar{W} = 0$$

where $\bar{W}$ is any vector orthogonal to the incident wavefront.

Now recalling that $A(z,z') = A_+$ for $z < z'$ and $A(z,z') = A_-$ for $z > z'$ we see that the relationships expressed by equations (59), (61), (69) and (70) can be combined into the following single statement: the matrix $A(z,z')$ projects every vector orthogonally onto the planes of the reflected or incident wavefront according as $z < z'$ or $z > z'$, respectively. Since $A(z,z')$ thereby maps a three dimensional space onto a two dimensional one, it must, by a well known theorem[18] in linear algebra, be singular.

This property of $A(z,z')$ enables us to verify immediately that the vectors $\bar{E}_R$ and $\bar{E}_T$ as given by formulas (43) and (45) actually lie in the planes of their respective wavefronts as required by Maxwell's equations; the geometrical significance of the singular character of the matrix $A(z,z')$ is thus made clear.

To conclude this section, we introduce a matrix S related to T by the equation

$$T = I + S \tag{71}$$

Evidently then, according to equation (67)

$$S_{ij} = \frac{ik}{2\cos\theta}\int_0^a e^{-ikz'\cos\theta}\,\bar{u}_i \cdot B(z')\,\bar{E}_{(j)}(z')\,dz' \qquad \begin{matrix}(i=1,2)\\(j=1,2)\end{matrix} \tag{72}$$

It will prove more natural to obtain the variational formulas for the elements S_{ij} than for T_{ij}

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[18] Birkhoff and Maclane, A Survey of Modern Algebra, p. 209.

The resulting [illegible] We see that the only [illegible] can be [illegible] jects every vector [illegible] vectors [illegible] maps a [illegible] space [illegible] known [illegible]

[illegible] vectors ξ_1 and ξ_2 [illegible] of [illegible] matrix [illegible] each other.

[illegible] the equation

(1) [illegible]

[illegible]

(2) [illegible]

[illegible] the [illegible]

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10. Birkhoff and MacLane, A Survey of Modern Algebra, p. 209.

5. The Variational Formulas

We are finally in a position to derive the variational formulas for the elements of the matrices R and S. The starting point of the derivation is the construction of a certain variational form involving inner products of elements in an abstract linear space; we shall prove that this form is stationary with respect to small variations in the elements on which it depends if these variations take place in the neighborhood of two particular elements of the space. Then, by constructing an explicit realization of the abstract space, we shall set up a correspondence between the elements of the latter and the vector functions of our original problem. Insertion of these vector functions into the variational form will lead to the desired stationary formulas for R_{ij} and S_{ij}.

For convenience, we recall briefly the definitions of linear space, inner product, and linear operation[19,20]. In all of our considerations of the abstract space, small Greek letters denote elements of the space, small Roman letters denote ordinary complex numbers, and capital Roman letters denote linear operations.

A "linear space" is a set of elements with the following properties:

1) For every pair of elements ζ_1 and ζ_2 of the space, there exists a commutative and associative operation indicated by + such that $\zeta_1 + \zeta_2$ is also an element of the space.

2) Corresponding to any complex number c and element ζ, there is a distributive and associative operation denoted by $c\zeta$, such that $c\zeta$ belongs to the space, and $1\cdot\zeta = \zeta$.

3) There exists a null element in the space, denoted by 0, with the properties $\zeta + 0 = 0$, $c\cdot 0 = 0$, $0\cdot\zeta = 0$

The "inner product" of two elements in the linear space, denoted by (ζ_1,ζ_2), is a complex number having the properties that $(c\zeta_1,\zeta_2) = c^*(\zeta_1,\zeta_2)$,

$$(\zeta_1+\zeta_2,\zeta_3) = (\zeta_1,\zeta_3) + (\zeta_2,\zeta_3),\ (\zeta_1,\zeta_2) = (\zeta_2,\zeta_1)^*,\ (\zeta_1,\zeta_1) > 0$$

for $\zeta_1 \neq 0$, $(\zeta_1,\zeta_1) = 0$ for $\zeta_1 = 0$. The asterisk denotes complex conjugate.

A "linear operator", K, in the linear space is a correspondence which maps each element ζ of the space on an element $K\zeta$ of the space in such fashion that for all elements ζ and ξ and all complex numbers c and d, $K(c\zeta + d\xi) = cK\zeta + dK\xi$.

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[19] Birkhoff and Maclane, Op.cit., Chapter 7.

[20] Stone, M.H., Linear Trans. in Hilbert Space, Chapters 1 and 2.

We define the linear operator which is "adjoint" to K and which we denote by K', through the property that for all elements ζ and ξ, $(K'\zeta, \xi) = (\zeta, K\xi)$.

Finally, if L and K are two linear operators, then by the operator LK applied to an element ζ we mean $L(K\zeta)$. The adjoint of LK is then easily shown to be K'L'.

Now let ζ and ζ' be any two variable elements of the linear space, let α and α' be two fixed elements of the space, let L and K be linear operators and L' and K' their respective adjoints. Consider the form[21]

$$73)\qquad h[\zeta,\zeta'] = \frac{(L'\zeta',\alpha)(\alpha',L\zeta)}{(\zeta',LKL\zeta)}$$

From its definition, h is evidently a complex number. We shall show that h is stationary with respect to arbitrary small variations in ζ and ζ' about the elements η and η', where η satisfies the equation

$$74)\qquad \alpha = KL\eta$$

and η' satisfies the equation

$$75)\qquad \alpha' = K'L'\eta'$$

More precisely, let η and η' satisfy equations (74) and (75). Then, if for ζ and ζ' we write

$$76)\qquad \zeta = \eta + s\xi \qquad\qquad \zeta' = \eta' + s'\xi'$$

where s and s' are complex numbers, and ξ and ξ' arbitrary elements of the linear space, we shall prove

$$77)\qquad \frac{\partial}{\partial s} h\left[\eta + s\xi, \eta' + s'\xi'\right]_{s'=0} = 0 \qquad \text{for all } s$$

$$78)\qquad \frac{\partial}{\partial s'} h\left[\eta + s\xi, \eta' + s'\xi'\right]_{s=0} = 0 \qquad \text{for all } s'$$

Equations (77) and (78) express a property of h which is stronger than the usual stationary one. Normally equation (77) would hold only for $s = 0$ and equation (78) only for $s' = 0$.

To prove equation (77), we first note that from the definition of the adjoint operator, it follows that $(\zeta',LKL\zeta) = (L'K'L'\zeta',\zeta)$. Intro-

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[21] Lax, M., Phys. Rev. vol. 78, p. 306, May 1, 1950.

ducing this equality into equation (73) and setting $s' = 0$, we get

$$79) \qquad h\left[\eta + s\xi\, , \eta'\right] = \frac{(L'\eta', \alpha)(\alpha', L(\eta + s\xi))}{(L'K'L'\eta', \eta + s\xi)} \, .$$

But according to equation (75), $K'L'\eta' = \alpha'$ so that $L'K'L'\eta' = L'\alpha'$. Therefore,

$$80) \qquad (L'K'L'\eta', \eta + s\xi) = (L'\alpha', \eta + s\xi) = (\alpha', L(\eta + s\xi))$$

Substitution of the last term on the right of equation (80) for the denominator in equation (79) gives

$$81) \qquad h\left[\eta + s\xi\, , \eta'\right] = (L'\eta', \alpha)$$

and since the right side of equation (81) is independent of s, equation (77) is thereby proved.

The verification of equation (78) is accomplished in similar fashion. We set $s = 0$ in equation (73) to obtain

$$82) \qquad h\left[\eta\, , \eta' + s'\xi'\right] = \frac{(L'(\eta' + s'\xi'), \alpha)(\alpha', L\eta)}{(\eta' + s'\xi', LKL\eta)}$$

Using equation (74), the denominator in equation (82) is seen to obey the equations

$$83) \qquad (\eta' + s'\xi', LKL\eta) = (\eta' + s'\xi', L\alpha) = (L'(\eta' + s'\xi'), \alpha)$$

from which equation (82) becomes

$$84) \qquad h\left[\eta\, , \eta' + s'\xi'\right] = (\alpha', L\eta)$$

which verifies equation (78).

The value of $h[\zeta, \zeta']$ at the stationary point $\zeta = \eta$, $\zeta' = \eta'$, may be written in several forms which we set down for future reference

$$85) \qquad h\left[\eta\, , \eta'\right] = \frac{(L'\eta', \alpha)(\alpha', L\eta)}{(\eta', LKL\eta)} = (\alpha', L\eta) = (L'\eta', \alpha)$$

The next step in the direction of applying the foregoing stationary property of h is to construct an explicit representation of the abstract linear space. For the representors we shall choose the set of all three component complex vector functions of the real variable z, continuous over the interval $0 \leq z \leq a$. Since the z dependent parts of the electromagnetic field vectors in our original problem belong to this set, the appropriateness of our choice is evident. It is also immediately clear that this set of vector functions actually constitutes a linear space obeying postulates (1), (2), and (3) if to the abstract + operation of postulate (1)

we make correspond the operation of ordinary vector addition, and to the abstract multiplication operation of postulate (2) we make correspond the multiplication of a vector by a complex scalar.

To complete the correspondence between the abstract space and its representation, we must define what we mean by inner product, linear operation, and adjoint operation as applied to the vector functions. We denote the correspondence between any abstract entity and its explicit representation by an arrow $\rightarrow$. Then if $\zeta_1 \rightarrow \bar{Z}_{(1)}(z)$ and $\zeta_2 \rightarrow \bar{Z}_{(2)}(z)$, the inner product (ζ_1, ζ_2) is defined by the correspondence

$$(\zeta_1, \zeta_2) \rightarrow \int_0^a \bar{Z}_{(1)}^*(z) \cdot \bar{Z}_{(2)}(z)\, dz \tag{86}$$

where the asterisk means complex conjugate and the dot indicates ordinary scalar product of two vectors. It is trivial to verify that the inner product as thus defined possesses all the properties attributed to the abstract inner product.

Next we define two kinds of linear operations as applied to the vector functions. If $\zeta \rightarrow \bar{Z}(z)$, then the linear operation $M\zeta$ is defined by

$$M\zeta \rightarrow M(z)\, \bar{Z}(z) \tag{87}$$

where $M(z)$ is a square matrix of rank 3 whose elements are continuous complex functions of z over the interval $0 \leq z \leq a$. Naturally $M(z)\, \bar{Z}(z)$ indicates matrix multiplication with $\bar{Z}(z)$ regarded as a column matrix. The linear operation $N\zeta$, on the other hand, we define as

$$N\zeta \rightarrow \int_0^a N(z,z')\, \bar{Z}(z')\, dz' \tag{88}$$

where $N(z,z')$ is a matrix similar to M, but whose elements are continuous complex functions of both z and z' over the region $0 \leq z \leq a$, $0 \leq z' \leq a$, except for a possible finite jump at $z = z'$.

It is equally trivial to verify that the operations defined by correspondences (87) and (88) possess the defining property of the abstract linear operation.

We define the operation adjoint to M by

$$M'\zeta \rightarrow M'(z)\, \bar{Z}(z) \tag{89}$$

where the matrix $M'(z)$ is obtained from $M(z)$ by interchange of rows and columns in the latter and replacement of all elements by their complex

conjugates. We shall verify that if $\xi \rightarrow \bar{V}(z)$, then

90) $$(M'\zeta, \xi) = (\zeta, M\xi)$$

in accordance with the defining property of the adjoint. We have

91) $$(M'\zeta, \xi) \rightarrow \int_0^a [M'(z)\bar{Z}(z)]^* \cdot \bar{V}(z)dz = \int_0^a \sum_{i=1}^{3}\Big[\sum_{j=1}^{3} M'_{ij}(z)Z_j(z)\Big]^* V_i(z)dz$$

$$= \int_0^a \sum_{i=1}^{3}\sum_{j=1}^{3} M'^*_{ij}(z)\, Z^*_j(z)V_i(z)dz = \int_0^a \sum_{j=1}^{3}\sum_{i=1}^{3} Z^*_j(z)M_{ji}(z)V_i(z)dz$$

$$= \int_0^a \sum_{j=1}^{3} Z^*_j(z)\, \Big[\sum_{i=1}^{3} M_{ji}(z)V_i(z)\Big]dz = \int_0^a \bar{Z}^*(z) \cdot M(z)\, \bar{V}(z)\, dz \rightarrow (\zeta, M\xi)$$

which proves equation (90).

Finally, the operation adjoint to N is defined by

92) $$N'\zeta \rightarrow \int_0^a N'(z,z')\bar{Z}(z)dz$$

where the matrix $N'(z,z')$ bears the same relationship to $N(z,z')$ as $M'(z)$ does to $M(z)$. It should be noted that the integration in (92) is with respect to z. Again letting $\zeta \rightarrow \bar{Z}(z)$ and $\xi \rightarrow \bar{V}(z)$ we shall verify that

93) $$(N'\zeta, \xi) = (\zeta, N\xi).$$

94) $$(N'\zeta, \xi) \rightarrow \int_0^a dz' \int_0^a N'^*(z,z')\, \bar{Z}^*(z) \cdot \bar{V}(z')dz$$

$$= \int_0^a dz' \int_0^a \sum_{i=1}^{3}\sum_{j=1}^{3} N'^*_{ij}(z,z')\, Z^*_j(z)V_i(z')dz = \int_0^a dz' \int_0^a \sum_{j=1}^{3} Z^*_j(z) \sum_{i=1}^{3} N_{ji}(z,z')V_i(z')dz$$

$$= \int_0^a \sum_{j=1}^{3} Z^*_j(z)dz \int_0^a \sum_{i=1}^{3} N_{ji}(z,z')\, V_i(z')dz' = \int_0^a dz\, \bar{Z}^*(z) \cdot \int_0^a N(z,z')\, \bar{V}(z')\, dz' \rightarrow (\zeta, N\xi)$$

where the reversal of the order of integration is justified because all of the functions are continuous except for the possible finite jump in $N_{ji}(z,z')$ at $z=z'$. Thus equation (93) is proved.

With the correspondence between the abstract linear space and the linear space of the vector functions now completed, we can proceed directly to the derivation of the variational formulas for R_{ij} and S_{ij}. To begin with, consider the variational form $h[\zeta,\zeta']$ given by equation (73). Let us permit the fixed element α to be either one of two particular fixed elements α_j ($j = 1,2$) and the fixed element α' to be either one of two particular fixed elements α_i' ($i = 1,2$). Similarly, let the variable element ζ be either one of two variable elements ζ_j($j = 1,2$) and the variable element ζ' either one of two variable elements ζ_i' ($i = 1,2$). Introducing these possibilities into equation (73) we get

$$95)\quad h[\zeta_j,\zeta_i'] = \frac{(L'\zeta_i',\alpha_j)(\alpha_i',L\zeta_j)}{(\zeta_i',\ LKL\zeta_j)} \qquad \begin{matrix}(j = 1,2)\\(i = 1,2)\end{matrix}$$

thereby obtaining four different variational forms of the type defined by equation (73), corresponding to the four possible subscript pairs i,j ($i = 1,2$), ($j = 1,2$). The variational property of $h[\zeta_j,\zeta_i']$ may evidently be expressed by the statement that $h[\zeta_j,\zeta_i']$ is stationary with respect to small variations of ζ_j and ζ_i' about the elements η_j and η_i' respectively, where η_j satisfies the equation

$$96)\qquad \alpha_j = KL\eta_j \qquad (j = 1,2)$$

and η_i' the equation

$$97)\qquad \alpha_i' = K'\ L'\eta_i' \qquad (i = 1,2)$$

For convenience, let us denote by h_{ij} the value of $h[\zeta_j,\zeta_i']$ at its stationary point. According to equation (85), we have

$$98)\ h_{ij} = h[\eta_j,\eta_i'] = \frac{(L'\eta_i',\alpha_j)(\alpha_i',L\eta_j)}{(\eta_i',LKL\eta_j)} = (\alpha_i',\ L\eta_j) = (L'\eta_i',\alpha_j) \qquad \begin{matrix}(j = 1,2)\\(i = 1,2)\end{matrix}$$

We now make two observations. The first is that linear operations and inner products of the type defined by correspondences (86),(87), and (88) appear in the integral equation formulation of our electromagnetic problem and in the integral formulas (66) and (72) for R_{ij} and S_{ij}. The second,

which we shall verify below, is that these linear operations together with certain electromagnetic vectors can be put into correspondence with the abstract operations and elements occurring in equation (98) in such fashion that when the abstract operations and elements are replaced by their electromagnetic counterparts, the resulting value of h_{ij} is R_{ij} or S_{ij}, depending on the precise correspondence chosen.

We verify the above assertion first for R_{ij}, obtaining for the latter the desired variational formula as the end result of the verification. The necessary correspondences are set up in a systematic way as follows: starting with the integral formula (66) for R_{ij}, which we reproduce here

99) $$R_{ij} = \frac{ik}{2\cos\Theta}\int_0^a e^{ikz\cos\Theta}\,\bar{u}_i' \cdot B(z)\,\bar{E}_{(j)}(z)\,dz$$

we recognize that according to the definition of inner product given by correspondence (86), we shall have

100) $$(\alpha_i' \,,\, L\eta_j) = R_{ij}$$

if the following correspondences are made:

101) $$\alpha_i' \rightarrow \bar{u}_i'\, e^{-ikz\cos\Theta}$$

102) $$L \rightarrow \frac{ik}{2\cos\Theta}\,B(z)$$

103) $$\eta_j \rightarrow \bar{E}_{(j)}(z)$$

Now from its definition, $\bar{E}_{(j)}(z)$ satisfies the integral equation

104) $$\bar{u}_j\, e^{ikz\cos\Theta} = \left[\, I + C(z)\right]\bar{E}_{(j)}(z) - \frac{ik}{2\cos\Theta}\int_0^a e^{ik|z-z'|\cos\Theta} A(z,z')B(z')\bar{E}_{(j)}(z')\,dz'$$

$(j = 1,2)$

Therefore, in view of correspondence (103), we can make equations (96) correspond to equation (104) if

105) $$\alpha_j \rightarrow \bar{u}_j\, e^{ikz\cos\Theta}$$

and if

$$106)\quad KL \rightarrow \left[I + C(z) \right] - \frac{ik}{2\cos\theta} \int_0^a e^{ik|z-z'|\cos\theta} A(z,z')\, B(z') \;....dz'$$

whence, from correspondence (102) we obtain

$$107)\quad K \rightarrow \left[I + C(z) \right] \frac{2\cos\theta}{ik} B^{-1}(z) - \int_0^a e^{ik|z-z'|\cos\theta} A(z,z') \;....dz'$$

There remains to establish a correspondence between η_1' and some vector. Since η_1' satisfies equation (97), we set up the integral equation corresponding to equation (97) and define the vector corresponding to η_1' as the solution of this integral equation.

For this purpose, we must form the adjoint operations L' and K' according to the defining correspondences (89) and (92).

From correspondence (102) we have

$$108)\quad L' \rightarrow - \frac{ik}{2\cos\theta} B'(z)$$

while correspondence (107) gives

$$109)\quad K' \rightarrow - \frac{2\cos\theta}{ik} B'^{-1} \left[I + C'(z) \right] - \int_0^a e^{-ik|z-z'|\cos\theta} A(z',z) ...dz' \;^{*}$$

where to obtain the first term on the right of (109) from the corresponding term in (107) we have used the fact that $(B^{-1})' = (B')^{-1}$ for any non-singular matrix B plus the fact that if F and G are any two matrices, then $(FG)' = G'F'$.

Combining correspondences (108) and (109), we get

$$110)\quad K'L' \rightarrow \left[I + B'^{-1}(z)\, C'(z) B'(z) \right] + \frac{ik}{2\cos\theta} \int_0^a e^{-ik|z-z'|\cos\theta} A(z',z) B'(z') ...dz'$$

The first term on the right of (110) can be simplified. From the definition of C(z), we see that C'(z) is a matrix whose first two columns are zero and whose last column equals that of B'(z). Therefore, we have at once

$$(111)\quad B'^{-1}(z) C'(z) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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* $A'(z',z) = A(z',z)$ since the matrix A is real and symmetric.

From equation (111) there follows

$$112)\quad B'^{-1}(z)\,C'(z)\,B'(z) = \begin{pmatrix} 000 \\ 000 \\ 001 \end{pmatrix} B'(z) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ B'_{31} & B'_{32} & B'_{33} \end{pmatrix}$$

and since $C(z)$ was defined as the matrix $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ B_{31} & B_{32} & B_{33} \end{pmatrix}$, it is clear that the last term on the right of equation (112) bears the same relationship to $B'(z)$ as $C(z)$ does to $B(z)$. Denoting this last term by $\tilde{C}(z)$, the final form of $K'L'$ becomes

$$113)\quad K'L' \longrightarrow \left[\, I + \tilde{C}(z) \,\right] + \frac{ik}{2\cos\theta} \int_0^a e^{-ik|z-z'|\cos\theta} A(z',z)\, B'(z') \,\ldots\, dz' .$$

Now making use of correspondences (97), (101), and (113), and denoting by $\bar{E}'_{(i)}(z)$ the vector obeying the correspondence

$$114)\qquad \eta'_i \longrightarrow \bar{E}'_{(i)}(z)$$

it follows that $\bar{E}'_{(i)}(z)$ satisfies the integral equation

$$115)\quad \bar{u}'_i\, e^{-ikz\cos\theta} = \left[\, I + \tilde{C}(z) \,\right] \bar{E}'_{(i)}(z) + \frac{ik}{2\cos\theta} \int_0^a e^{-ik|z-z'|\cos\theta} A(z',z)\, B'(z')\, \bar{E}'_{(i)}(z')\, dz'$$

$$(i = 1,2)$$

which has a form precisely analogous to that of the integral equation (104). Equation (115) is called the "adjoint" equation to (104). Its physical interpretation will be given in Section 6, which deals with the reciprocity theorem referred to in the Introduction.

We have thus set up the electromagnetic counterparts of each of the abstract elements and operations appearing in equation (98), as well as the counterparts of equations (96) and (97). Combining equations (98) and (100) we have

$$116)\qquad R_{ij} = \frac{(L'\eta'_i,\ \alpha_j)(\alpha'_i,\ L\,\eta_j)}{(\eta'_i,\ LKL\,\eta_j)}$$

and replacing each of the abstract entities in formula (116) by its electromagnetic counterpart, we obtain immediately the final variational formula for R_{ij}

$$117)\quad R_{ij} = \frac{ik}{2\cos\theta}\,\frac{\left[\int_0^a e^{ikz\cos\theta}\,\bar{u}_j\cdot B'^*(z)\bar{E}'^*_{(i)}(z)dz\right]\left[\int_0^a e^{ikz\cos\theta}\,\bar{u}'_i\cdot B(z)\bar{E}_{(j)}(z)\,dz\right]}{\int_0^a \bar{E}'^*_{(i)}(z)\cdot B(z)\left[I+C(z)\right]\bar{E}_{(j)}(z)dz - \frac{ik}{2\cos\theta}\int_0^a\int_0^a e^{ik|z-z'|\cos\theta}\,\bar{E}'^*_{(i)}(z)\cdot B(z)A(z,z')B(z')\bar{E}_{(j)}(z')dz'dz}$$

$(i = 1,2)$

$(j = 1,2)$

Because of the stationary character of the form $h[\zeta_j, \zeta_i']$ and the correspondence we have established between the abstract space and the space of vector functions, we can sum up the properties of formula (117) in the following way: if $\bar{E}_{(j)}(z)$ satisfies the integral equation (104) and $\bar{E}_{(i)}'(z)$ satisfies the adjoint equation (115), then of course formula (117) gives the exact value of R_{ij}; on the other hand, if for $\bar{E}_{(j)}(z)$ and $\bar{E}_{(i)}'(z)$ we substitute approximate trial functions into formula (117), then the formula will be stationary with respect to small departures of the trial functions from the true solutions to the integral equations (104) and (115), respectively. Naturally, it is the second property which makes formula (117) more accurate than formula (99) for the purpose of approximate calculation of R_{ij}.

The procedure for deriving the variational formulas for S_{ij} is exactly like the preceding one save that the electromagnetic counterparts of α_i' and η_i' must be changed. Reproducing here the integral formula (72) for S_{ij},

$$118) \qquad S_{ij} = \frac{ik}{2\cos\theta}\int_0^a e^{-ikz\cos\theta}\,\bar{u}_i \cdot B(z)\,\bar{E}_{(j)}(z)\,dz$$

we see that we shall have

$$119) \qquad (\alpha_i', L\eta_j) = S_{ij}$$

if

$$120) \qquad \alpha_i' \rightarrow \bar{u}_i\, e^{ikz\cos\theta}$$

while correspondences (102) and (103) remain unchanged.

If we again define $\bar{E}_{(i)}'(z)$ by the correspondence

$$121) \qquad \eta_i' \rightarrow \bar{E}_{(i)}'(z)$$

where η_i' is the solution to equation (97), then it follows from equation (97) and correspondences (113) and (120) that $\bar{E}_{(i)}'(z)$ now satisfies the integral equation

$$122) \quad \bar{u}_i\, e^{ikz\cos\theta} = \left[I + \tilde{C}(z)\right]\bar{E}_{(i)}'(z) + \frac{ik}{2\cos\theta}\int_0^a e^{-ik|z-z'|\cos\theta} A(z',z)B'(z')\bar{E}_{(i)}'(z')\,dz'$$

Equation (122) is the adjoint to equation (104). Thus the adjoint integral equation associated with S_{ij} differs from that associated with R_{ij}, viz., equation (115), only in the "incident wave" term on the left. The physical interpretation of equation (122) will also be given in Section 6.

Combining equations (98) and (119) we have

$$(123) \qquad S_{ij} = \frac{(L'\eta_i', \alpha_j)(\alpha_i', L\eta_j)}{(\eta_i', LKL\,\eta_j)}$$

Again replacing each of the abstract entities in formula (123) by its electromagnetic counterpart, we obtain the variational formula for S_{ij}

$$124) \quad S_{ij} = \frac{ik}{2\cos\theta} \frac{\left[\int_0^a e^{ikz\cos\theta}\,\bar{u}_j \cdot B'^*(z)\bar{E}'^*_{(i)}(z)dz\right]\left[\int_0^a e^{-ikz\cos\theta}\,\bar{u}_i \cdot B(z)\,\bar{E}_{(j)}(z)\,dz\right]}{\int_0^a \bar{E}'^*_{(i)}(z)\cdot B(z)[I + C(z)]\,\bar{E}_{(j)}(z)dz - \frac{ik}{2\cos\theta}\int_0^a\int_0^a e^{ik|z-z'|\cos\theta}\,\bar{E}'^*_{(i)}(z)\cdot B(z)A(z,z')B(z')\bar{E}_{(j)}(z')dz'dz}$$

Naturally, T_{ij} is obtained from formula (124) by the relationship

$$125) \qquad T_{ij} = \delta_{ij} + S_{ij}$$

With the derivation of formulas (117) and (124), we have completed the task of deducing variational expressions for the elements of the transmission and reflection matrices of our original electromagnetic problem.

6. The Reciprocity Theorem

In the previous section we set up certain correspondences between the abstract entities of the linear space and the vectors and operations of the electromagnetic problem, and we showed that the correspondences could be so chosen that the inner product $(\alpha_i', L\eta_j)$ equaled R_{ij} or S_{ij}, whichever was desired. We did not, however, inquire into the resulting electromagnetic significance of the analogous inner product $(L'\eta_i', \alpha_j)$ and we wish to do so now. Considering first the case where the correspondences are those associated with R_{ij}, we shall show that $(L'\eta_i', \alpha_j)$ can be interpreted as the jith element of the reflection matrix in a new electromagnetic problem whose relationship to the original problem is a very simple one. We shall show further that one of the equalities in equation (98), namely,

$$(\alpha_i' , L\eta_j) = (L'\eta_i', \alpha_j) \tag{126}$$

is actually the expression of an interesting reciprocity theorem which connects the reflection matrix of the new or "reciprocal" problem with the reflection matrix of the original problem.

We deduce the statement of the reciprocal problem in the following way: according to correspondences (105), (108), and (114), together with definition (86) of the inner product, we have

$$(L'\eta_i' , \alpha_j) = \frac{ik}{2\cos\theta}\int_0^a e^{ikz\cos\theta}\,\bar{u}_j\cdot B'^*(z)\,\bar{E}'^*_{(i)}(z)\,dz \tag{127}$$

where $\bar{E}'_{(i)}(z)$ satisfies the adjoint integral equation (115).

A comparison of expression (99) with the right side of equation (127) suggests that the latter may be interpreted as the jith element of the reflection matrix for a reciprocal problem in which the scattering medium has a susceptibility matrix $B'^*(z)$ while the electric field within the medium is $\bar{E}'^*_{(i)}(z)$. The exact statement of the reciprocal problem is inferred from the integral equation satisfied by $\bar{E}'^*_{(i)}(z)$. Since $\bar{E}'_{(i)}(z)$ is the solution to the adjoint integral equation (115), it follows from the linearity of this equation that $\bar{E}'^*_{(i)}(z)$ satisfies the complex conjugate

of the adjoint integral equation, i.e.,

$$128)\quad \bar{u}'_i e^{ikz\cos\theta} = [I+\tilde{C}^*(z)]\bar{E}'^*_{(i)}(z) - \frac{ik}{2\cos\theta}\int_0^a e^{ik|z-z'|\cos\theta} A(z',z)B'^*(z)\bar{E}'^*_{(i)}(z')dz'$$

$$(i = 1,2)$$

This is the integral equation for the reciprocal problem. To interpret it, we compare it with the integral equation for the original problem, viz.,

$$129)\quad \bar{u}_j e^{ikz\cos\theta} = [I+C(z)]\bar{E}_{(j)}(z) - \frac{ik}{2\cos\theta}\int_0^a e^{ik|z-z'|\cos\theta} A(z,z')B(z')\bar{E}_{(j)}(z')\,dz'$$

$$(j = 1,2)$$

The first point of difference to be noted between equations (128) and (129) is in the susceptibility matrices. It will be recalled that $B'(z)$ was derived from $B(z)$ by transposition of rows and columns in the latter and replacement of all elements by their complex conjugates; $B'^*(z)$ is therefore just the ordinary transpose of $B(z)$. Moreover $\tilde{C}(z)$ was defined as having its first two rows zero and its third row equal to the conjugate of the third column of $B(z)$; therefore $\tilde{C}^*(z)$ bears the identical relationship to $B'^*(z)$ as $C(z)$ does to $B(z)$. We conclude that in the reciprocal problem, the scattering medium has a susceptibility matrix which is the transpose of the matrix characterizing the original medium. We call the new medium the "transposed medium".

The second point of difference between the two integral equations is that the matrix $A(z,z')$ in equation (129) has been replaced by $A(z',z)$ in equation (128). Referring to equations (26) and (27), which define $A(z,z')$, we see that interchanging z and z' is equivalent to substituting $-\theta$ for θ in the matrix. Inspection of the remaining θ terms in equations (128) and (129) then shows that as far as the θ dependence is concerned, equation (128) is obtainable from (129) simply by replacing θ in the latter by $-\theta$. We conclude that in the reciprocal problem, the angle of incidence is $-\theta$.

Finally, it is clear from equation (128) that the incident wave vector amplitude for the reciprocal problem is $\bar{u}_i'$. This observation completes the definition of the reciprocal problem.

Now in Section 4, we derived the integral formula (66) or (99) for R_{ij}. The same derivation obviously applies at once to the reciprocal problem; in fact if we denote by R' the reflection matrix for the reciprocal problem we have

$$130) \qquad R_{ji}' = \frac{ik}{2\cos\theta}\int_0^a e^{ikz\cos\theta}\,\bar{u}_j \cdot B'^*(z)\,\bar{E}_{(i)}'^*(z)\,dz$$

Combining equations (100), (126), (127), and (130), there results

$$131) \qquad R_{ij} = R_{ji}'$$

Equation (131) constitutes the reciprocity theorem for the reflection matrix. The verbal statement of this theorem is as follows: in the original problem, let an incident wave of vector amplitude $\bar{u}_j$ and angle of incidence θ excite a reflected wave whose amplitude component in the $\bar{u}_i'$ direction is R_{ij}; then an incident wave of vector amplitude $\bar{u}_i'$ which impinges on the transposed medium at an angle of incidence $-\theta$, will excite a reflected wave whose amplitude component in the $\bar{u}_j$ direction is also equal to R_{ij}.

The reciprocity theorem for the transmission matrix is derived in similar fashion. Using correspondences (105), (108), and (121), we obtain

$$132) \qquad (L'\eta_i', \alpha_j) = \frac{ik}{2\cos\theta}\int_0^a e^{ikz\cos\theta}\,\bar{u}_j\cdot B'^*(z)\,\bar{E}_{(i)}'^*(z)\,dz$$

where $\bar{E}_{(i)}'(z)$ now satisfies the adjoint equation (122). From the form of expression (132), it is again reasonable to define the reciprocal problem as being governed by the complex conjugate of the adjoint equation. Thus the integral equation for the reciprocal problem is now

$$133) \; \bar{u}_i e^{-ikz\cos\theta} = [I+\tilde{C}^*(z)]\bar{E}_{(i)}'^*(z) - \frac{ik}{2\cos\theta}\int_0^a e^{ik|z-z'|\cos\theta} A(z',z)B'^*(z')\bar{E}_{(i)}'^*(z')dz'$$

For comparison, we also write the equation of the original problem once more

$$134) \; \bar{u}_j e^{ikz\cos\theta} = [I+C(z)]\bar{E}_{(j)}(z) - \frac{ik}{2\cos\theta}\int_0^a e^{ik|z-z'|\cos\theta} A(z,z')B(z')\bar{E}_{(j)}(z')dz'$$

From equation (133) it is clear that in the transmission matrix reciprocal problem the original scattering medium is again to be replaced by the transposed medium. However, the complete specification of the reciprocal problem requires slightly more detailed consideration of the conjugate adjoint equation than was necessary in the case of the reflection matrix.

Consider first the form of $\bar{E}'^{*}_{(i)}(z)$ in the region $z > a$. Since in this region $\tilde{C}^{*}(z) = 0$ and $|z-z'| = z-z'$, equation (133) gives

$$135)\quad \bar{E}'^{*}_{(i)}(z) = \bar{u}_i e^{-ikz\cos\theta} + e^{ikz\cos\theta}\frac{ik}{2\cos\theta}\int_0^a e^{-ikz'\cos\theta} A_{+}B'^{*}(z')\bar{E}'^{*}_{(i)}(z')dz' \qquad z > a$$

Thus in the region $z > a$, $\bar{E}'^{*}_{(i)}(z)$ represents a downward moving incident wave of amplitude $\bar{u}_i$ impinging on the scattering medium from above, plus an upward moving reflected wave. From the fact that $A(z,z')$ in equation (134) is replaced by $A(z',z)$ in equation (133), it is easily inferred that the x dependence of the field vectors in the reciprocal problem must be $e^{-ik\,x\sin\theta}$. We conclude that in the transmission matrix reciprocal problem, the incident wave strikes the transposed medium from above at an angle of incidence $\pi + \theta$ with the positive z direction, where θ is the angle of incidence in the original problem.

In the region $z < 0$, $\tilde{C}^{*}(z) = 0$ and $|z-z'| = z'-z$. Equation (133) then becomes

$$136)\quad \bar{E}'^{*}_{(i)}(z) = \bar{u}_i e^{-ikz\cos\theta} + e^{-ikz\cos\theta}\frac{ik}{2\cos\theta}\int_0^a e^{ikz'\cos\theta} A_{-}B'^{*}(z')\bar{E}'^{*}_{(i)}(z')dz' \qquad z < 0$$

Therefore in the region $z < 0$, $\bar{E}'^{*}_{(i)}(z)$ represents a downward moving transmitted wave whose propagation direction makes an angle $\pi + \theta$ with the positive z direction.

Now let T' denote the transmission matrix for the reciprocal problem. Taking the scalar product of equation (136) with $\bar{u}_j$ and dropping the factor $e^{-ikz\cos\theta}$, it follows from the definition of transmission matrix

given in Section (4) that

137) $$T'_{ji} = \delta_{ji} + \frac{ik}{2\cos\theta} \int_0^a e^{ikz'\cos\theta}\, \bar{u}_j \cdot A_- B'^*(z')\, \bar{E}'^*_{(i)}(z')\, dz'$$

which in view of equation (63) and the symmetry of A_-, reduces to

138) $$T'_{ji} = \delta_{ji} + \frac{ik}{2\cos\theta} \int_0^a e^{ikz'\cos\theta}\, \bar{u}_j \cdot B'^*(z')\, \bar{E}'^*_{(i)}(z')\, dz'$$

If we define the matrix S' through the relation

139) $$T' = I + S'$$

we have from equation (138)

140) $$S'_{ji} = \frac{ik}{2\cos\theta} \int_0^a e^{ikz'\cos\theta}\, \bar{u}_j \cdot B'^*(z')\, \bar{E}'^*_{(i)}(z')\, dz'$$

Combining equations (119), (126), (132), and (140), we obtain

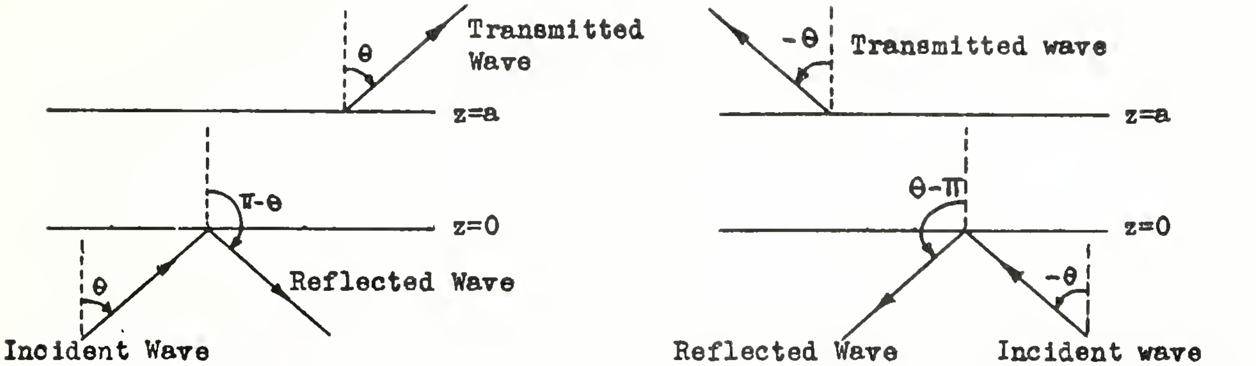
141) $$S_{ij} = S'_{ji}$$

or equivalently

142) $$T_{ij} = T'_{ji}$$

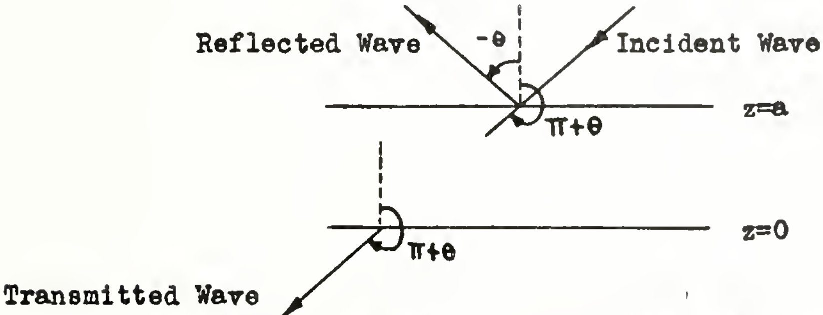
Equation (142) expresses the reciprocity theorem for the transmission matrix. In words it may be stated as follows: in the original problem, let an incident wave of vector amplitude $\bar{u}_j$ and angle of incidence θ excite a transmitted wave whose amplitude component in the $\bar{u}_i$ direction is T_{ij}; then an incident wave of vector amplitude $\bar{u}_i$ impinging from above on the transposed medium at an angle of incidence $\pi + \theta$ with the positive z direction, will excite a transmitted wave in the region $z < 0$ whose amplitude component in the $\bar{u}_j$ direction is also equal to T_{ij}.

The relationship between the original problem and the two reciprocal problems is indicated graphically in Fig. 2. The lines with arrows represent the propagation directions of the respective waves.



Original Problem

Reciprocal Problem for Reflection Matrix



Reciprocal Problem for Transmission Matrix

Fig. 2

We conclude with an example which will illustrate the physical relation between the original scattering medium and the transposed medium. Assume that the region $0 \leq z \leq a$ is filled with a mixture of electrons and positive ions. The electron and ion densities are equal at every point, depend only on z, and naturally vanish at $z = 0$ and $z = a$. We assume that there is also present in the region some magnetic field $\bar{H}_e$ which is in general a function of z but constant in time. Finally, we suppose that the mass of an ion is very much greater than that of an electron. Such a medium (known as a magneto-ionic medium) is generally used to represent the earth's ionosphere, $\bar{H}_e$ then being the earth's magnetic field.

Now let $N(z)$ be the electron or ion density function, e and m respectively the charge and mass of the electron, ω the angular frequency of a time harmonic electromagnetic wave impinging on the medium, and $\nu(z)$ the number of collisions per second which a given electron makes with ions or atoms. Then the susceptibility matrix of such a medium can be shown[22]

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[22] Wilkes, M.V. Proc.Roy.Soc. A, vol. 189, p. 130, 1947. See equation (2.2) of Wilkes' paper.

to be

143)
$$B(z) = \frac{X}{(1+iZ)\left[(1+iZ)^2-(Y_1^2+Y_2^2+Y_3^2)\right]}\begin{pmatrix} Y_1^2-(1+iZ)^2 & iY_3(1+iZ)+Y_1Y_2 & Y_1Y_3-iY_2(1+iZ) \\ Y_2Y_1-iY_3(1+iZ) & Y_2^2-(1+iZ)^2 & iY_1(1+iZ)+Y_2Y_3 \\ iY_2(1+iZ)+Y_3Y_1 & Y_3Y_2-iY_1(1+iZ) & Y_3^2-(1+iZ)^2 \end{pmatrix}$$

where $X(z) = \frac{N(z)e^2}{m\omega^2\epsilon_o}$, $Y_i(z) = \frac{eH_{e_i}(z)}{m\omega}$ $(i = 1,2,3)$

and $Z(z) = \frac{\nu(z)}{\omega}$. H_{e_i} is the x,y, or z component of $\bar{H}_e$.

Inspection of equation (143) shows at once that transposition of the matrix B(z) is equivalent to replacing Y_1, Y_2, and Y_3 by $-Y_1$, $-Y_2$ and $-Y_3$ respectively. Therefore if the original scattering medium is a magneto-ionic medium of the type just defined, the transposed medium differs from the original only in that the magnetic field $\bar{H}_e$ is reversed in direction at every point.

7. Conclusion

Schwinger's integral equation-variational technique has been applied to the problem of the scattering of an elliptically polarized plane electromagnetic wave by a continuously stratified anisotropic medium. Variational formulas have been obtained for the elements of the reflection and transmission matrices of the problem. The efficacy of certain considerations in the theory of linear spaces with respect to construction of the variational expressions has been demonstrated. Finally, a certain symmetry property of the variational formulas has been used to prove a reciprocity theorem.

Appendix I

We prove that if $\bar{f}$ and $\bar{g}$ are solutions to Maxwell's equations in a region where $\mu = \mu_o$ and ϵ is a continuous but otherwise arbitrary matrix function of position, i.e.,

1a) $$\nabla \times \bar{f} - i\omega\mu_o\bar{g} = 0$$

2a) $$\nabla \times \bar{g} + i\omega\epsilon\bar{f} = 0 ,$$

then there always exists at least one vector function $\bar{\phi}$ satisfying the equation

3a) $$\nabla^2\bar{\phi} + k^2\bar{\phi} = -\frac{1}{\epsilon_o}(\epsilon - \epsilon_o I)\bar{f}$$

in the given region, and having the further property that

4a) $$\bar{f} = \nabla(\nabla \cdot \bar{\phi}) + k^2\bar{\phi}$$

5a) $$\bar{g} = -i\omega\epsilon_c \nabla \times \bar{\phi}$$

Taking the divergence of equation (1a), we get

6a) $$\nabla \cdot \bar{g} = 0$$

Since $\bar{g}$ is continuous with continuous first derivatives in the region where it satisfies equations (1a) and (2a), then by a fundamental theorem in vector analysis there exists a vector function $\bar{\phi}_o$ such that

7a) $$\bar{g} = -i\omega\epsilon_o \nabla \times \bar{\phi}_o$$

in the given region, where the constant factor $-i\omega\epsilon_o$ is added for convenience.

Substitution from equation (7a) into equation (1a) gives

8a) $$\nabla \times (\bar{f} - k^2\bar{\phi}_o) = 0$$

Thus the vector $\bar{f} - k^2\bar{\phi}_o$ is irrotational in the given region. By another fundamental theorem in vector analysis there must exist a scalar function q_o such that

9a) $$\bar{f} - k^2\bar{\phi}_o = -\nabla q_o$$

Let us write equation (2a) as

10a) $$\nabla \times \bar{g} + i\omega\epsilon_o\bar{f} = -i\omega(\epsilon - \epsilon_o I)\bar{f}$$

Substituting from equations (7a) and (9a) into equation (10a), we obtain

11a) $$-i\omega\epsilon_o \nabla \times \nabla \times \bar{\phi}_o - i\omega\epsilon_o \nabla q_o + i\omega\epsilon_o k^2\bar{\phi}_o = -i\omega(\epsilon - \epsilon_o I)\bar{f}$$

which after division by $i\omega\epsilon_o$ becomes

12a) $$-\nabla\times\nabla\times\bar{\phi}_o - \nabla q_o + k^2\bar{\phi}_o = -\frac{1}{\epsilon_o}(\epsilon - \epsilon_o I)\,\bar{f}$$

Equation (12a) is a necessary condition on any pair of functions $\bar{\phi}_o$ and q_o which are related to $\bar{f}$ and $\bar{g}$ by equations (7a) and (9a).

Now let $\bar{\phi}_o$ and q_o represent one particular pair of functions satisfying equations (7a), (9a), and (12a); then the functions $\bar{\phi}$ and q defined by

13a) $$\bar{\phi} = \bar{\phi}_o + \nabla p$$

14a) $$q = q_o + k^2 p$$

where p is an arbitrary differentiable scalar, also satisfy equations (7a), (9a), and therefore (12a), i.e.,

15a) $$\bar{g} = -i\omega\epsilon_o\,\nabla\times\bar{\phi}$$

16a) $$\bar{f} = -\nabla q + k^2\bar{\phi}$$

17a) $$-\nabla\times\nabla\times\bar{\phi} - \nabla q + k^2\bar{\phi} = -\frac{1}{\epsilon_o}(\epsilon - \epsilon_o I)\,\bar{f}$$

Since p is arbitrary, let us choose it to be a solution of the equation

18a) $$\nabla^2 p + k^2 p = -\nabla\cdot\bar{\phi}_o - q_o$$

Equation (18a) may be written as

19a) $$\nabla\cdot(\bar{\phi}_o + \nabla p) = -(q_o + k^2 p)$$

or

20a) $$\nabla\cdot\bar{\phi} = -q$$

in view of equations (13a) and (14a).

Substitution of the value of q given by equation (20a) into equation (17a) gives

21a) $$-\nabla\times\nabla\times\bar{\phi} + \nabla(\nabla\cdot\bar{\phi}) + k^2\bar{\phi} = -\frac{1}{\epsilon_o}(\epsilon - \epsilon_o I)\,\bar{f}$$

which in rectangular coordinates becomes

22a) $$\nabla^2\bar{\phi} + k^2\bar{\phi} = -\frac{1}{\epsilon_o}(\epsilon - \epsilon_o I)\,\bar{f}$$

Under the same substitution, equation (16a) becomes

23a) $$\bar{f} = \nabla(\nabla\cdot\bar{\phi}) + k^2\bar{\phi}\ .$$

Equations (22a), (23a), and (15a) agree with equations (3a), (4a), and (5a) respectively. Therefore the function $\bar{\Phi}$ defined by equation (13a), where p satisfies equation (18a), obeys equations (3a), (4a), and (5a). The existence of the desired $\bar{\Phi}$ is thereby proved.

Appendix II

In this appendix carry out the evaluation of the term

$$1b)\quad \left[\nabla(\nabla\cdot\ ) + k^2\right] \frac{e^{ikx\sin\theta}}{2i\epsilon_0 k\cos\theta} \int_0^a e^{ik|z-z'|\cos\theta}\,\bar{P}(z')\,dz'$$

(see equation 24), where the differential operator in expression (1b) applies of course to the variables x and z. Beginning with the operation $(\nabla\cdot\ ) = \underline{i}\frac{\partial}{\partial x} + \underline{k}\frac{\partial}{\partial z}$ and ignoring the constant term $\frac{1}{2i\epsilon_0 k\cos\theta}$, we have

$$2b)\quad (\underline{i}\frac{\partial}{\partial x} + \underline{k}\frac{\partial}{\partial z})\cdot e^{ikx\sin\theta}\int_0^a e^{ik|z-z'|\cos\theta}\,\bar{P}(z')\,dz'$$

$$= (\underline{i}\frac{\partial}{\partial x} + \underline{k}\frac{\partial}{\partial z})\cdot e^{ikx\sin\theta}\left[\int_0^z e^{ik(z-z')\cos\theta}\bar{P}(z')dz' + \int_z^a e^{ik(z'-z)\cos\theta}\bar{P}(z')dz'\right]$$

The operations on the right of equation (2b) result in

$$3b)\quad \underline{i}\frac{\partial}{\partial x}\cdot\left[e^{ikx\sin\theta}\int_0^z e^{ik(z-z')\cos\theta}\bar{P}(z')dz'\right]$$

$$= ik\sin\theta\, e^{ikx\sin\theta}\int_0^z e^{ik(z-z')\cos\theta}\,P_1(z')dz' \quad *$$

$$4b)\quad \underline{i}\frac{\partial}{\partial x}\cdot\left[e^{ikx\sin\theta}\int_z^a e^{ik(z'-z)\cos\theta}\bar{P}(z')dz'\right]$$

$$= ik\sin\theta\, e^{ikx\sin\theta}\int_z^a e^{ik(z'-z)\cos\theta}\,P_1(z')\,dz'$$

$$5b)\quad \underline{k}\frac{\partial}{\partial z}\cdot\left[e^{ikx\sin\theta}\int_0^z e^{ik(z-z')\cos\theta}\bar{P}(z')\,dz'\right]$$

$$= ik\cos\theta\, e^{ikx\sin\theta}\int_0^z e^{ik(z-z')\cos\theta}P_3(z')dz' + e^{ikx\sin\theta}P_3(z)$$

---

* P_1 and P_3 are the x and z components of $\bar{P}$, respectively.

$$6b)\quad \underline{k}\,\frac{\partial}{\partial z}\cdot\left[e^{ikx\sin\theta}\int_z^a e^{ik(z'-z)\cos\theta}\,\bar{P}(z')\,dz'\right]$$

$$= -\,ik\cos\theta\, e^{ikx\sin\theta}\int_z^a e^{ik(z'-z)\cos\theta}\,P_3(z')dz' - e^{ikx\sin\theta}\,P_3(z)$$

Adding these equations, we get

$$7b)\quad \nabla\cdot\left[e^{ikx\sin\theta}\int_0^a e^{ik|z-z'|\cos\theta}\,\bar{P}(z')dz'\right]$$

$$= ik\sin\theta\, e^{ikx\sin\theta}\left[\int_0^z e^{ik(z-z')\cos\theta}P_1(z')dz' + \int_z^a e^{ik(z'-z)\cos\theta}P_1(z')dz'\right]$$

$$+\, ik\cos\theta\, e^{ikx\sin\theta}\left[\int_0^z e^{ik(z-z')\cos\theta}\,P_3(z')dz' - \int_z^a e^{ik(z'-z)\cos\theta}P_3(z')dz'\right]$$

Next, we must take the gradient of the right side of equation (7b), i.e., we must operate on it with $(\underline{i}\,\frac{\partial}{\partial x} + \underline{k}\,\frac{\partial}{\partial z})$. The x differentiation gives after a little rearranging

$$8b)\quad k^2 e^{ikx\sin\theta}\int_0^z e^{ik(z-z')\cos\theta}\left[-\sin^2\theta\, P_1(z') - \sin\theta\cos\theta\, P_3(z')\right]\underline{i}\,dz'$$

$$+\, k^2\, e^{ikx\sin\theta}\int_z^a e^{ik(z'-z)\cos\theta}\left[-\sin^2\theta\, P_1(z') + \sin\theta\cos\theta\, P_3(z')\right]\underline{i}\,dz'$$

while from the z differentiation there results

$$9b)\quad k^2\, e^{ikx\sin\theta}\int_0^z e^{ik(z-z')\cos\theta}\left[-\sin\theta\cos\theta\, P_1(z') - \cos^2\theta\, P_3(z')\right]\underline{k}\,dz'$$

$$+\, ik\, e^{ik\,x\sin\theta}\left[\sin\theta\, P_1(z)\underline{k} + \cos\theta\, P_3(z)\,\underline{k}\right]$$

$$+k^2 e^{ikx\sin\theta}\int_z^a e^{ik(z'-z)\cos\theta}\left[\sin\theta\cos\theta\, P_1(z') - \cos^2\theta\, P_3(z')\right]\underline{k}\,dz'$$

$$+\, ik\, e^{ikx\sin\theta}\left[-\sin\theta\, P_1(z)\,\underline{k} + \cos\theta\, P_3(z)\,\underline{k}\right]$$

The complete operation $\left[\nabla(\nabla\cdot\ ) + k^2\right]$ is now formed by adding to the sum of expressions (8b) and (9b) the term

$$10b)\quad k^2 e^{ik\,x\,\sin\theta}\int_0^a e^{ik|z-z'|\cos\theta}\,\bar{P}(z')\,dz'$$

$$= k^2 e^{ikx\,\sin\theta}\int_0^z e^{ik(z-z')\cos\theta}\left[P_1(z')\underline{i} + P_2(z')\underline{j} + P_3(z')\,\underline{k}\right] dz'$$

$$+ k^2 e^{ikx\,\sin\theta}\int_z^a e^{ik(z'-z)\cos\theta}\left[P_1(z')\underline{i} + P_2(z')\underline{j} + P_3(z')\,\underline{k}\right] dz'$$

The result of this addition may be arranged as follows:

$$11b)\quad k^2 e^{ikx\,\sin\theta}\int_0^z e^{ik(z-z')\cos\theta}\left[\cos^2\theta\, P_1(z') + 0\cdot P_2(z') - \sin\theta\cos\theta\, P_3(z')\right] dz'\ \underline{i}$$

$$+ k^2 e^{ik\,x\sin\theta}\int_0^z e^{ik(z-z')\cos\theta}\left[0\cdot P_1(z') + P_2(z') + 0\cdot P_3(z')\right] dz'\ \underline{j}$$

$$+ k^2 e^{ikx\,\sin\theta}\int_0^z e^{ik(z-z')\cos\theta}\left[-\sin\theta\cos\theta\, P_1(z') + 0\cdot P_2(z') + \sin^2\theta P_3(z')\right] dz'\ \underline{k}$$

$$+ k^2 e^{ikx\sin\theta}\int_z^a e^{ik(z'-z)\cos\theta}\left[\cos 2\ \theta\ P_1(z') + 0\cdot P_2(z') + \sin\theta\cos\theta P_3(z')\right] dz'\ \underline{i}$$

$$+ k^2 e^{ikx\sin\theta}\int_z^a e^{ik(z'-z)}\cos\theta\left[0\cdot P_1(z') + P_2(z') + 0\cdot P_3(z')\right] dz'\ \ \underline{j}$$

$$+ k^2 e^{ikx\sin\theta}\int_z^a e^{ik(z'-z)}\cos\theta\left[\sin\theta\,\mathrm{vos}\,\theta\, P_1(z') + 0\cdot P_2(z') + \sin^2\theta P_3(z')\right] dz'\ \underline{k}$$

$$+ 2ik\ e^{ikx\sin\theta}\cos\theta\, P_3(z)\ \underline{k}$$

$$= k^2 e^{ikx\sin\theta}\int_0^a e^{ik|z-z'|\cos\theta} A(z,z')\ \bar{P}(z')\ dz' + 2ik\ e^{ik\,x\,\sin\theta}\cos\theta\, P_3(z)\ \underline{k}$$

where the matrix $A(z,z')$ is defined by equations (26) and (27).

Finally we multiply equation (11b) by $\dfrac{1}{2i\epsilon_0 k\cos\theta}$ to obtain

$$12b)\ \left[\nabla(\nabla\cdot\ ) + k^2\right]\frac{e^{ikx\sin\theta}}{2i\epsilon_0 k\cos\theta}\int_0^a e^{ik|z-z'|\cos\theta}\,\bar{P}(z')dz'$$

$$= -\frac{ike^{ikx\sin\theta}}{2\epsilon_0\cos\theta}\int_0^a e^{ik|z-z'|\cos\theta}A(z,z')\bar{P}(z')dz' + \frac{1}{\epsilon_0}e^{ikx\sin\theta}P_3(z)\,\underline{k}$$

which is equation (25).

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